The Davenport Constant of Finite Abelian Groups

Thesis submitted for the degree of
Doctor of Philosophy in Mathematics by

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Declaration of Authorship

I Aleen Sheikh hereby declare that this thesis and the work presented in it is entirely my own. Where I have consulted the work of others, this is always clearly stated.

Signed:

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Abstract

Let G be a finite abelian group. The *Davenport constant* of G, denoted $D(G)$, is the smallest integer d such that every sequence over G of length d has a non-empty zero-sum subsequence. The problem of finding the Davenport constant of an arbitrary finite abelian group is a well-known problem in combinatorial number theory.

It is known that

$$D(G) \geq 1 + d^*(G),$$

where $d^*(G)$ is a certain constant that is computed using the invariant factor decomposition of G. There was a conjecture that this bound is always tight, but counterexamples are now known for many groups G of rank 4 or more. However, the conjecture has been established for many classes of groups, in particular Olson proved in 1969 that $D(G) = 1 + d^*(G)$ when G has rank at most 2. Whether the conjecture holds when G has rank 3 is still an open problem.

The main results of the thesis are as follows. We prove the equality $D(G) = 1 + d^*(G)$ for $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$, the smallest group of rank 3 where this equality was not known. We provide a detailed proof of a result of Bhowmik and Schlage-Puchta from 2007, which shows that $D(G) = 1 + d^*(G)$ holds for groups G of the form $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$. Our proof removes some of the obscurities in their original approach. Finally, we establish new upper bounds on $D(G)$ in terms of $d^*(G)$, including a general quadratic upper bound and a linear upper bound in the case when $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d}$.

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Chapter 1

Introduction

1.1 Fundamental concepts

Throughout this text, all groups are assumed to be non-trivial, finite and abelian and shall be written additively. We begin with some definitions that form the foundations of what is known as the theory of *zero-sum sequences* over finite abelian groups.

Definition 1.1.1. A *sequence* $S = s_1 \cdots s_n$ of *length* $n \in \mathbb{N}$ over a group G is an unordered collection of elements $s_1, \dots, s_n$ of G where repetition is allowed.

Definition 1.1.2. A *subsequence* of a sequence $S = s_1 \cdots s_n$ over a group G is a sequence over G of the form $s_{i_1} \cdots s_{i_j}$ where $i_1, \dots, i_j$ are pairwise distinct elements in $\{1, \dots, n\}$.

Definition 1.1.3. The *value* $|S|$ of a sequence $S = s_1 \cdots s_n$ over a group G is defined to be $s_1 + \cdots + s_n \in G$.

Definition 1.1.4. A sequence S over a group G is called *zero-sum* if $|S| = 0_G$.

Example 1.1.5. Let $G = \mathbb{Z}_5$. Then $S = 0111224 = 0112421$ is an example of a sequence over G of length 7 such that $|S| = 1$. The subsequence $1112 = 1121$ of S is zero-sum.

Remark 1.1.6. The empty sequence is zero-sum.

Definition 1.1.7. The *Davenport constant* $D(G)$ of a group G is the smallest $d \in \mathbb{N}$ such that every sequence over G of length d has a non-empty zero sum subsequence.

Remark 1.1.8. It is widely known that Harold Davenport proposed the constant defined in Definition 1.1.7 in a conference in 1966. However, it is seldom mentioned that Kenneth Rogers proposed such a constant and its applications in algebraic number theory in 1963 (see [27]).

The following trivial upper bound on the Davenport constant shows that it is well defined.

Lemma 1.1.9. *For all finite abelian groups G, we have*

$$D(G) \leq |G|.$$

Proof. Let G be a group. We show that every sequence over G of length $|G|$ contains a non-empty zero-sum subsequence. Suppose $|G|= n$ and let $s_1 \dots s_n$ be a sequence over G. For each $k \in [1, n]$, define $b_k := |s_1 \cdots s_k|$. If $b_k = b_j$ for some $k < j$ then $|s_{k+1} \cdots s_j|= b_j - b_k = 0_G$. If b_k are all distinct then we have n distinct elements of G and so one of them, say $b_j = |s_1 \cdots s_j|$, must be 0_G. $\square$

In general, given an arbitrary finite abelian group G, there is no known efficient method for determining the Davenport constant of G. A broad aim of the thesis is to determine the Davenport constant for as many finite abelian groups as possible.

Definition 1.1.10. A sequence S over a group G is called *zero-sum free* if the only zero-sum subsequence of S is the empty sequence.

Remark 1.1.11. Note that

1. the maximal length of a zero-sum free sequence over a finite abelian group G is equal to $D(G) - 1$;

2. the Davenport constant is preserved under isomorphism.

We shall represent elements of the group $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ as r-tuples $(a_1, \ldots, a_r)$ where $a_i \in \mathbb{Z}_{n_i}$ for all $1 \leq i \leq r$, with component-wise addition as the group operation. Recall the *invariant factor decomposition* of a finite abelian group.

Theorem 1.1.12 (Corollary 10.38 in [29])**.** *For any non-trivial finite abelian group G, there exist unique parameters $1 < n_1 \mid \cdots \mid n_r \in \mathbb{N}$ such that*

$$G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}\,.$$

Definition 1.1.13. For $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$, we define

- $\operatorname{rank}(G) := r$;
- $d^*(G) := \sum_{i=1}^{r}(n_i - 1)$;
- $\exp(G) := n_r$.

Remark 1.1.14. For $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$, we have

$$d^*(G) = n_1 + \cdots + n_r - r \geq 2r - r = r$$

with equality if and only if $G \cong \mathbb{Z}_2^d$ for some $d \in \mathbb{N}$.

The constant $d^*(G)$ is significant in determining the value of the Davenport constant for many groups G as we will see in the next section.

1.2 Motivation

The motivation for the main results in the thesis stems from a trivial lower bound on the Davenport constant as we will now see.

Definition 1.2.1. The *union* of two non-empty sequences $S = s_1 \cdots s_m$ and $S' = s'_1 \cdots s'_n$ over a group G, denoted $S \cup S'$, is defined to be the sequence $s_1 \cdots s_m s'_1 \cdots s'_n$ over G.

Lemma 1.2.2. *For all finite abelian groups G, we have*

$$1 + d^*(G) \leq D(G).$$

Proof. It is sufficient to let $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ for some $1 < n_1 \mid \cdots \mid n_r$ and find a zero-sum free sequence over G of length $d^*(G)$. Consider the sequence $S = S_1 \cup \cdots \cup S_r$ where for each $i \in [1, r]$, S_i is the sequence over G consisting of $(n_i - 1)$ copies of e_i, where e_i denotes the r-tuple in G with 1 in the i-th position and 0 elsewhere. Then S is a zero-sum free sequence over G of length $d^*(G)$. □

After glancing over the sequence in the proof of Lemma 1.2.2, intuitively it may seem like it is not possible to conceive a zero-sum free sequence over a group G of length strictly greater than $d^*(G)$ (in fact, it is mentioned in [4] that this was conjectured by P.C. Baayen). This is the case for many classes of groups however, this is not the case for all groups. More precisely, the equality

$$D(G) = 1 + d^*(G) \tag{1.1}$$

holds for many classes of groups G but there also exist many classes of groups G for which $D(G) > 1 + d^*(G)$. There exist groups G with $D(G) > 1 + d^*(G)$ for which the precise value of the Davenport constant is known (for example, see [9] and [6]). There is no general conjecture regarding the precise value of the Davenport constant for an arbitrary finite abelian group.

Examples of classes of groups for which (1.1) holds include p-groups (see [23]) and groups with rank at most 2 (see [24]). An example of a class of groups G for which $D(G) > 1 + d^*(G)$ is $G \cong \mathbb{Z}_m \oplus \mathbb{Z}_n^2 \oplus \mathbb{Z}_{2n}$ where m and n are odd with $m \geq 3$ and $m|n$ (see [15]). In fact, it has been shown that for each $r \geq 4$ there exist infinitely many groups G of rank r for which $D(G) > 1 + d^*(G)$. The interesting, and unsolved, case is when the rank of the group is 3. The equality (1.1) has been shown to hold for many classes of groups of rank 3. However, it is not known whether the equality holds for all groups of rank 3. Some authors conjecture the following.

Conjecture 1.2.3 ([3], [13])**.** *The equality $D(G) = 1 + d^*(G)$ holds for all finite abelian groups G of rank* 3.

There are two main aims of the thesis. One of them is to prove Conjecture 1.2.3 for as many groups as possible. The other aim is to find upper bounds on $D(G)$ in terms of $d^*(G)$ for an arbitrary finite abelian group G. Given the lower bound on the Davenport constant in Lemma 1.2.2 it seems natural to enquire about such upper bounds. However, it is striking to discover that such upper bounds on the Davenport constant do not seem to have been discussed in previous literature.

There are three key results in the thesis. We determine the Davenport constant of the group $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$, which is the smallest abelian group of rank 3 for which the Davenport constant was unknown. More precisely, we show that the equality $D(G) = 1 + d^*(G)$ holds for $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$. In 2007, Bhowmik and Schlage-Puchta proved the equality $D(G) = 1 + d^*(G)$ for the class of groups $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ (see [3]). However, we believe that their proof contains some obscurities. We reconstruct their proof to produce a version which is less obscure. Finally, we find new upper bounds on $D(G)$ in terms of $d^*(G)$ for finite abelian groups G, including a general quadratic upper bound and a linear upper bound in the case when $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d}$.

The structure of the thesis is as follows. We start with a survey of the literature on the Davenport constant in Chapter 2. In Chapter 3 we present a result which compares the constants $d^*(G)$ and $d^*(H)$ where G is a finite abelian group with subgroup H. We reconstruct the proof of the equality $D(G) = 1 + d^*(G)$ for the class of groups $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ in Chapter 4. In Chapter 5 we present some partial results aimed towards showing the equality $D(G) = 1 + d^*(G)$ for the class of groups $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{2p}$ where p is an arbitrary prime number. In Chapter 6 we determine the Davenport constant of the group $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$. In Chapter 7 we prove a linear upper bound on $D(G)$ in terms of $d^*(G)$ when $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d}$. The proofs of some results in the thesis involve the use of computer programs which search for sequences with particular properties over the groups $\mathbb{Z}_3^3$ and $\mathbb{Z}_5^3$. These computer programs are detailed in Chapter 8. In Chapter 9 we present new general upper bounds on $D(G)$ in terms of $d^*(G)$ for finite abelian groups G. Finally, in Chapter 10 we briefly discuss some open problems relating to the Davenport constant.

Chapter 2

A survey of the Davenport constant

This chapter reviews previous literature on the Davenport constant with a focus on results that show the Davenport constant of a group meets the trivial lower bound of Lemma 1.2.2.

In Section 2.1 we list all finite abelian groups for which the trivial lower bound is previously known to be tight and briefly mention the groups for which the bound is known not to be tight. In Section 2.2 we find the smallest abelian group of rank 3 for which the trivial lower bound is not previously known to be tight.

2.1 The trivial lower bound

The aim of this section is to review finite abelian groups G for which the equality $D(G) = 1 + d^*(G)$ holds. We briefly mention groups for which the equality does not hold.

In the 1960s, the following results emerged about the value of the Davenport constant.

Theorem 2.1.1 ([23]). *Let G be an abelian p-group. Then $D(G) = 1 + d^*(G)$.*

Theorem 2.1.2 (Theorem 1 in [24]). *Let G be a finite abelian group of rank at most 2. Then $D(G) = 1 + d^*(G)$.*

As mentioned in Chapter 1, the authors of [3] and [13] conjecture that $D(G) = 1 + d^*(G)$ for all finite abelian groups G of rank 3. An example of a group of rank 3 for which this equality was proven quite early on is the following.

Theorem 2.1.3 (Lemma 1.1 in [5]). *The equality $D(G) = 1 + d^*(G)$ holds when $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6$.*

It is mentioned in [4] that in 1965 P.C. Baayen originally conjectured that the equality $D(G) = 1 + d^*(G)$ holds for all finite abelian groups G. However, the theorem below shows that by 1969 P.C. Baayen had found a counterexample.

Theorem 2.1.4 (Theorem 8.1 in [4]). *Let $G \cong \mathbb{Z}_2^{4k} \oplus \mathbb{Z}_{4k+2}$ for some $k \in \mathbb{N}$. Then $D(G) > 1 + d^*(G)$.*

The inequality $D(G) > 1 + d^*(G)$ is now known for lots of other classes of groups G. In fact, it is known that for each $r \geq 4$, there exist infinitely many groups G of rank r such that $D(G) > 1 + d^*(G)$. This is essentially proved in [15] by combining Lemma 1 and Theorem 3 in [15]:

Theorem 2.1.5 (Theorem 3 in [15]). *Let $G \cong \mathbb{Z}_m \oplus \mathbb{Z}_n^2 \oplus \mathbb{Z}_{2n}$ where m and n are odd with $m \geq 3$ and $m|n$. Then $D(G) > 1 + d^*(G)$.*

Another open conjecture relating to the Davenport constant is the following.

Conjecture 2.1.6 ([13]). *The equality $D(G) = 1+d^*(G)$ holds when $G \cong \mathbb{Z}_n^r$ where n and r are arbitrary positive integers.*

Next, we gather together an extensive list of finite abelian groups G for which $D(G) = 1 + d^*(G)$. To the best of our knowledge, this list covers all finite abelian groups G so far known for which the equality holds.

List 2.1.7 (Groups G for which it is known that $D(G) = 1 + d^*(G)$)**.**

1. p-groups G ([23]),
2. G with $\mathrm{rank}(G) \leq 2$ (Theorem 1 in [24]),
3. $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ where $d \in \mathbb{N}$ (Theorem 1 in [3]),
4. $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{3d} \oplus \mathbb{Z}_{3d}$ where $\gcd(d, 6) = 1$ (combine Theorem 5 in [2] and the result on page 3 in [25]),
5. $G \cong \mathbb{Z}_{3.2^t} \oplus \mathbb{Z}_{3.2^u} \oplus \mathbb{Z}_{3.2^v}$ where $v \geq u \geq t \geq 0$ (Corollary 1.5 in [5]),
6. $G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{4d}$ where $d \in \mathbb{N}$ (Theorem 4.1 in [28]),
7. $G \cong \mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{6d}$ where $d \in \mathbb{N}$ (Theorem 4.1 in [28]),
8. $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{p^n m}$ where p is a prime, $n \geq 2$ and $\gcd(m, p^n) = 1$ (Proposition 4.3 in [8]),
9. $G \cong \mathbb{Z}_{2p^t} \oplus \mathbb{Z}_{2p^u} \oplus \mathbb{Z}_{2p^v}$ where p is a prime and $v \geq u \geq t \geq 0$ (Corollary 4.3 in [4]),
10. $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2na} \oplus \mathbb{Z}_{2nb}$ where $n = 2^t 3^u 5^v 7^w$ for some $t, u, v, w \geq 0$, and either $a = 1$ and b is arbitrary, or $a = p^r$ and $b = p^s$ with p prime and $s \geq r \geq 0$ (Corollary 5.6 in [4]),
11. $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{6na} \oplus \mathbb{Z}_{6nb}$ where n, a, b are as in (10) (Corollary 1.5 in [5]),
12. $G \cong \mathbb{Z}_2^3 \oplus \mathbb{Z}_{2d}$ where $d \in \mathbb{N}$ ([1]),
13. $G \cong \mathbb{Z}_2^4 \oplus \mathbb{Z}_{2d}$ where $d \geq 70$ is even (Theorem 5.8 in [6]).

Remark 2.1.8. Each class of groups in List 2.1.7 contains at least one group which is not in any of the other classes.

2.2 The smallest unsolved case of rank 3

We can see from List 2.1.7 that there are various classes of groups G of rank 3 for which the equality $D(G) = 1 + d^*(G)$ is known to hold. As mentioned before, it is not known whether this equality holds for all groups of rank 3. In this section we determine the smallest group of rank 3 for which the equality has not yet been investigated. More precisely, we prove the following:

Theorem 2.2.1. *The smallest abelian group of rank* 3 *for which the Davenport constant is unknown is* $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$.

Remark 2.2.2. There are precisely three abelian groups of order less than 250 with rank strictly greater than 3 for which the Davenport constant is unknown; they are $\mathbb{Z}_2^4 \oplus \mathbb{Z}_{12}$, $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$ and $\mathbb{Z}_2^4 \oplus \mathbb{Z}_{14}$.

This theorem updates the list on page 14 in [5] (see Remark 2.2.6). In order to prove this theorem we shall need the following auxiliary results.

Lemma 2.2.3. *Let H be a finite abelian group such that $|H| = p_1^{k_1} \cdots p_t^{k_t}$ where $p_1, \ldots, p_t$ are distinct primes and $k_1, \ldots, k_t > 0$. Then* $\mathrm{rank}(H) \leq k$ *where $k := max\{k_1, \ldots, k_t\}$.*

Proof. Suppose $H \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ where $1 < n_1 \mid \cdots \mid n_r \in \mathbb{N}$. Pick a prime divisor p of n_1. Since $n_1 | \cdots | n_r$, we have that the multiplicity of p in $|H|$ is at least r. Hence, $k \geq r = \mathrm{rank}(H)$. □

Lemma 2.2.4 (Corollary of Theorem 2.14.3 in [18])**.** *Given distinct prime numbers $p_1, \ldots, p_t$ and integers $k_1, \ldots, k_t > 0$, there are precisely $p(k_1) \cdots p(k_t)$ abelian groups of order $p_1^{k_1} \cdots p_t^{k_t}$ where $p(n)$ denotes the number of partitions of an integer n.*

Example 2.2.5. Find all abelian groups of order 36: We have $36 = 2^2 3^2$. Using Theorem 1.1.12 we can list the following 4 groups of order 36: $\mathbb{Z}_{36}$, $\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$, $\mathbb{Z}_3 \oplus \mathbb{Z}_{12}$, and $\mathbb{Z}_6 \oplus \mathbb{Z}_6$. Since $2p(2) = 4$, we deduce by Lemma 2.2.4 that this list is complete up to isomorphism.

We can now prove the main theorem of this section.

Proof of Theorem 2.2.1. Define $G := \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$. We claim that the equality $D(G) = 1 + d^*(G)$ is unknown. To show this, it is sufficient to show that G does not belong to any of the classes of groups mentioned in List 2.1.7. We have that G does not belong to (1) as the order of G is not a prime power. We have that G does not belong to (2), (12), or (13) as $\mathrm{rank}(G) = 3$. We have that G does not belong to (3), (4), (5), (6), (7), (9), (10), or (11) as the invariant factor decomposition of the groups in these classes does not contain the factor $\mathbb{Z}_5$. Lastly, we have that G does not belong to (8) since $25 \nmid 10$. We now claim that the inequality $D(G) > 1 + d^*(G)$ is unknown. Indeed this is the case else G is a counterexample to Conjecture 1.2.3. Since it is widely known that Conjecture 1.2.3 has not yet been proven false, the claim follows. Hence $D(G)$ is unknown according to the literature.

Let H be an arbitrary abelian group of rank 3 and order at most 249. We show that $D(H) = 1 + d^*(H)$. Define

$$\begin{aligned} A := & [2,23] \cup [25,39] \cup [41,47] \cup [49,53] \cup \{55\} \cup \\ & [57,71] \cup [73,79] \cup [81,87] \cup [89,95] \cup [97,103] \cup \\ & [105,107] \cup [109,111] \cup [113,119] \cup [121,134] \cup [137,151] \cup \\ & [153,159] \cup \{161\} \cup [163,167] \cup [169,175] \cup [177,183] \cup \\ & [185,188] \cup [190,191] \cup [193,199] \cup [201,207] \cup [209,215] \cup \\ & [217,223] \cup [225,231] \cup [233,239] \cup [241,247] \cup \{249\}. \end{aligned}$$

Case (i): Suppose $|H| \in A$. Then either $|H|$ is a prime power or the multiplicity of the most frequently occurring prime in the prime factorisation of $|H|$ is at most 2. If the former holds then H belongs to (1) from List 2.1.7. If the latter holds then Lemma 2.2.3 implies $\mathrm{rank}(H) < 3$ which is a contradiction.

Case (ii): Suppose $|H| \notin A$. Then using Lemma 2.2.4 and Theorem 1.1.12 we construct a table consisting of each possible isomorphism class for H, and the class in List 2.1.7 to which H belongs. See Table 2.1.

$\|H\|$	Possibilities for H	Class to which H belongs
24	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$	10 (take $n = a = 1$ and $b = 3$)
40	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{10}$	10 (take $n = a = 1$ and $b = 5$)
48	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$	10 (take $n = a = 1$ and $b = 6$)
54	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6$	3 (take $d = 2$)
56	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{14}$	10 (take $n = a = 1$ and $b = 7$)
72	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{18}$ $\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_6$	10 (take $n = a = 1$ and $b = 9$) 10 (take $n = 3$ and $a = b = 1$)
80	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{20}$	10 (take $n = a = 1$ and $b = 10$)
88	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{22}$	10 (take $n = a = 1$ and $b = 11$)
96	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{24}$ $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$	10 (take $n = a = 1$ and $b = 12$) 10 (take $n = 2$, $a = 1$ and $b = 3$)
104	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{26}$	10 (take $n = a = 1$ and $b = 13$)
108	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{12}$ $\mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_6$	3 (take $d = 4$) 11 (take $n = a = b = 1$)
112	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{28}$	10 (take $n = a = 1$ and $b = 14$)
120	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{30}$	10 (take $n = a = 1$ and $b = 15$)
135	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{15}$	3 (take $d = 5$)
136	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{34}$	10 (take $n = a = 1$ and $b = 17$)
152	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{38}$	10 (take $n = a = 1$ and $b = 19$)
160	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{40}$ $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{20}$	10 (take $n = a = 1$ and $b = 20$) 10 (take $n = 2$, $a = 1$ and $b = 5$)
162	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{18}$	3 (take $d = 6$)
168	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{42}$	10 (take $n = a = 1$ and $b = 21$)
176	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{44}$	10 (take $n = a = 1$ and $b = 22$)
184	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{46}$	10 (take $n = a = 1$ and $b = 23$)
189	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{21}$	3 (take $d = 7$)
192	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{48}$ $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{24}$ $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$	10 (take $n = a = 1$ and $b = 24$) 10 (take $n = 2$, $a = 1$ and $b = 6$) 6 (take $d = 3$)

$\lvert H\rvert$	Possibilities for H	Class to which H belongs
200	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{50}$	10 (take $n = a = 1$ and $b = 25$)
	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$	10 (take $n = 5$ and $a = b = 1$)
208	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{52}$	10 (take $n = a = 1$ and $b = 26$)
216	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{54}$	10 (take $n = a = 1$ and $b = 27$)
	$\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{18}$	10 (take $n = 3$, $a = 1$ and $b = 3$)
	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{24}$	3 (take $d = 8$)
	$\mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{12}$	11 (take $n = a = 1$ and $b = 2$)
	$\mathbb{Z}_6^3$	7 (take $d = 1$)
224	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{56}$	10 (take $n = a = 1$ and $b = 28$)
	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{28}$	10 (take $n = 2$, $a = 1$ and $b = 7$)
232	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{58}$	10 (take $n = a = 1$ and $b = 29$)
240	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{60}$	10 (take $n = a = 1$ and $b = 30$)
248	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{62}$	10 (take $n = a = 1$ and $b = 31$)

Table 2.1: Groups of rank 3 of order at most 249 which are not p-groups

Now it remains to show that the Davenport constant of all abelian groups different from G of order 250 is known. By Lemma 2.2.4 and Theorem 1.1.12 there are precisely two abelian groups order 250 other than G: they are $\mathbb{Z}_{250}$ and $\mathbb{Z}_5 \oplus \mathbb{Z}_{50}$. As the rank of both of these groups is at most 2, their Davenport constant is known by (2) in List 2.1.7. This completes the proof.

□

Remark 2.2.6. Page 14 in [5] lists the 13 abelian groups G with $\lvert G\rvert \leq 500$ and $\mathrm{rank}(G) = 3$ for which it was unknown whether $D(G) = 1 + d^*(G)$. Using (3) and (6) from List 2.1.7 we can update and shorten the list in [5] to the following:

G	$\lvert G\rvert$
$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$	250
$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{15}$	375
$\mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{18}$	486
$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{20}$	500
$\mathbb{Z}_5 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$	500

Chapter 3

A result about $d^*(H)$ for subgroups H

Given a group $G \cong \bigoplus_{i=1}^{r} \mathbb{Z}_{m_i}$ where $1 < m_1 \mid \cdots \mid m_r$ and given a subgroup H of G, it is clear from the definition that $d^*(H) \leq d^*(G)$ if $H \cong \bigoplus_{i \in I} \mathbb{Z}_{m_i}$ for some subset $I \subset \{1, \ldots, r\}$. In this chapter we show that the inequality remains intact even when H is not of the previously described form. More precisely we prove the following result:

Theorem 3.1. *Let G be a finite abelian group and H a subgroup of G. Then*

$$d^*(H) \leq d^*(G)$$

where equality holds if and only if $H = G$.

In order to prove this result we need the following theorem.

Theorem 3.2 (Theorem 3.22 in [22])**.** *Let $G \cong \bigoplus_{i=1}^{r} \mathbb{Z}_{m_i}$ for some integers $1 < m_1 \mid \cdots \mid m_r$. If H is a subgroup of G then $H \cong \bigoplus_{i=1}^{t} \mathbb{Z}_{n_i}$ where $t \leq r$, $n_i | n_{i+1}$ for all $1 \leq i \leq t-1$ and $n_i \mid m_{r-t+i}$ for all $1 \leq i \leq t$.*

Proof of Theorem 3.1. Let $G \cong \bigoplus_{i=1}^{r} \mathbb{Z}_{m_i}$ and $H \cong \bigoplus_{i=1}^{t} \mathbb{Z}_{n_i}$ for some integers $1 < m_1 \mid \cdots \mid m_r$ and $1 < n_1 \mid \cdots \mid n_t$. By Theorem 3.2 we have $t \leq r$.

This means that $t = r - \alpha$ for some $0 \le \alpha \le r - 1$. Moreover, Theorem 3.2 tells us that $n_i \mid m_{r-t+i}$ for all $1 \le i \le t$ which implies

$$m_{\alpha+i} \ge n_i$$

for all $1 \le i \le t$. Hence

$$d^*(G) = m_1 + \cdots + m_r - r \ge m_1 + \cdots + m_\alpha + n_1 + \cdots + n_t - r.$$

We have

$$m_1 + \cdots + m_\alpha \ge \alpha$$

which implies

$$m_1 + \cdots + m_\alpha + n_1 + \cdots + n_t - r \ge n_1 + \cdots + n_t - t.$$

Noting that $d^*(H) = n_1 + \cdots + n_t - t$, we find that $d^*(H) \le d^*(G)$. Now suppose $d^*(H) = d^*(G)$. This implies

$$m_1 + \cdots + m_\alpha = \alpha.$$

Therefore $\alpha = 0$ and hence $t = r$. From this we deduce that

$$m_1 + \cdots + m_r = n_1 + \cdots + n_r.$$

Therefore, since $m_i \ge n_i$ for all i, we deduce that $m_i = n_i$ for all i, and hence $G = H$. This completes the proof. $\square$

Definition 3.3. Given a non-empty sequence $S = s_1 \cdots s_n$ over a group G, define the *group generated by* S to be

$$\langle S \rangle := \langle s_1, \ldots, s_n \rangle.$$

We can deduce the following corollary from Theorem 3.1.

Corollary 3.4. *Let S be a sequence over a group G such that $\langle S \rangle = H$ for some subgroup H of G with $D(H) = 1 + d^*(H)$. If the length of S is at least $1 + d^*(G)$ then S is not zero-sum free.*

Proof. Using Theorem 3.1, note that

$$D(H) = 1 + d^*(H) \leq 1 + d^*(G).$$

Hence S is a sequence over H of length at least $D(H)$. This means that S is not zero-sum free by the definition of the Davenport constant. $\square$

Chapter 4

The Davenport constant of $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$

4.1 Motivation

In this chapter we reconstruct the proof of the equality $D(G) = 1 + d^*(G)$ for $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ where d is an arbitrary positive integer. This equality was originally proved by Gautami Bhowmik and Jan-Christoph Schlage-Puchta in [3]. There are a handful of places in their proof where we are unable to convince ourselves of the detail of the argument. In particular, we believe that their proof only explicitly deals with the case when $\gcd(d, 6) = 1$. Since there are extra complications when $\gcd(d, 6) \neq 1$, we believe the general case should be written down in detail. Furthermore, there is at least one statement in the their proof which is incorrect (see Remark 4.3.11). We follow the original proof but modify some aspects with new notation, rephrase some concepts, and replace some preliminary material with new results to produce a version of the proof which we hope convinces the reader that $D(\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}) = 1 + d^*(\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d})$ for all $d \in \mathbb{N}$.

The structure of this chapter is as follows. In Section 4.2 we present preliminary results applicable to an arbitrary finite abelian group that we need to reconstruct the proof. In Section 4.3 we present preliminary results relating specifically to the group $\mathbb{Z}_3^3$ that we need to reconstruct the proof. In Section 4.4 we present the reconstructed proof.

4.2 Preparatory material

In this section we present some preliminary material to be used later in the chapter. These results all hold for arbitrary finite abelian groups.

Definition 4.2.1. Let $S_1, \ldots, S_t$ be non-empty subsequences of a sequence $S = s_1 \cdots s_n$ over a group G. For each $i \in \{1, \ldots, t\}$, let $S_i = s_{i_1} \cdots s_{i_{j_i}}$ for pairwise distinct elements $i_1, \ldots, i_{j_i}$ in $\{1, \ldots, n\}$. We say $S_1, \ldots, S_t$ are *disjoint* if the collection

$$\{i_1, \ldots, i_{j_i}\}_{i \in [1,t]}$$

contains no repeated elements.

Definition 4.2.2. Given groups G and T and a map $\phi : G \longrightarrow T$ we define

- the *image* under ϕ of a non-empty sequence $S = s_1 \cdots s_n$ over G to be the sequence $\phi(S) := \phi(s_1) \cdots \phi(s_n)$ over T;

- a *T-zero-sum sequence* with respect to ϕ to be a non-empty sequence S over G such that $|\phi(S)| = 0_T$.

Given G, T and ϕ as in Definition 4.2.2, we may omit the reference to ϕ when talking about a T-zero-sum sequence with respect to ϕ if ϕ is clear from the context.

Lemma 4.2.3. *Let G be a finite abelian group and H a subgroup of G. Define $T := G/H$ and $d := D(H)$. Let S be a sequence over G such that S contains d disjoint T-zero-sum subsequences with respect to the canonical homomorphism $\phi : G \longrightarrow T$. Then S is not zero-sum free.*

Proof. Let $S_1, \dots, S_d$ be d disjoint T-zero-sum subsequences of S. Then for each $i \in [1, d]$ we have

$$0_G + H = |\phi(S_i)| = \phi(|S_i|) = |S_i| + H.$$

Hence $U := |S_1| \cdots |S_d|$ is a sequence of length d over H. Since $d = D(H)$, we deduce that U contains a non-empty zero-sum subsequence $|S_{i_1}| \cdots |S_{i_j}|$. Now it remains to note that $S_{i_1} \cup \dots \cup S_{i_j}$ is a non-empty zero-sum subsequence of S. $\square$

Proposition 4.2.4. *Let H be a finite abelian group of order $d \geq 2$ and let $S = s_1 \cdots s_{d-1}$ be a zero-sum free sequence of length $d - 1$ over H. Then*

$$s_1 = \dots = s_{d-1} = h$$

for some $h \in H$. In particular, $H = \langle h \rangle$.

Proof (derived from the proof of Proposition 1.7 in [4]). Suppose for a contradiction that there exist $i, j \in [1, d-1]$ such that $s_i \neq s_j$. Without loss of generality suppose $i = 1$ and $j = 2$. Consider the following $d - 1$ elements of H: s_1, $-s_2$, $s_1 + \sum_{k=3}^{k=k'} s_i$ where $3 \leq k' \leq d - 1$. Since S is zero-sum free, these $d - 1$ elements are pairwise distinct and non-zero and hence form the set $H \backslash \{0_H\}$. Now consider the element $s_1 - s_2 \in H$. If $s_1 - s_2 = s_1$ then $s_2 = 0_H$. If $s_1 - s_2 = -s_2$ then $s_1 = 0_H$. If $s_1 - s_2 = s_1 + \sum_{k=3}^{k=k'} s_i$ for some $3 \leq k' \leq d - 1$ then $s_2 + \sum_{k=3}^{k=k'} s_i = 0_H$. All of these cases contradict the assumption that S is zero-sum free. Therefore $s_1 - s_2 \notin H \backslash \{0_H\}$. However this contradicts the assumption that $s_1 \neq s_2$. Hence $s_1 = \dots = s_{d-1} = h$ for some $h \in H$.

As S is zero-sum free, we have that $h \neq 0_H$, $2h \neq 0_H$, $\dots$, $(d-2)h \neq 0_H$, $(d-1)h \neq 0_H$. So h generates H. This completes the proof. $\square$

Corollary 4.2.5. *Let G be a finite abelian group and H a non-trivial subgroup of G. Define $T := G/H$ and $d := |H|$. Let S be a zero-sum free sequence over G containing $d-1$ disjoint T-zero-sum subsequences $S_1, \dots, S_{d-1}$ with respect to the canonical homomorphism $\phi : G \longrightarrow T$. Then H is cyclic and*

$$|S_1| = \cdots = |S_{d-1}| = h$$

for some generator h of H.

Proof. For each $i \in [1, d-1]$ we have

$$0_G + H = |\phi(S_i)| = \phi(|S_i|) = |S_i| + H.$$

Hence $U := |S_1| \cdots |S_{d-1}|$ is a sequence of length $d-1$ over H. Observe that U is zero-sum free else S is not zero-sum free. The corollary now follows from Proposition 4.2.4. □

4.3 Preliminary results about $\mathbb{Z}_3^3$

In this section we present the preliminary results relating specifically to the group $\mathbb{Z}_3^3$.

Lemma 4.3.1 (Lemma 3 in [3]). *Let $S = s_1 \cdots s_5$ be a sequence of 5 pairwise distinct elements of $\mathbb{Z}_3^3$ without a non-empty zero-sum subsequence of length at most 3. Then there exists a subsequence $s_i s_j s_k$ of S such that $s_i + s_j = s_k$.*

Proof (derived from the proof of Lemma 3 in [3]). Suppose for a contradiction that the assertion in the statement of the lemma does not hold. Then, viewing $\mathbb{Z}_3^3$ as a 3-dimensional vector space over $\mathbb{Z}_3$, we find that the elements s_1, s_2 and s_3 are linearly independent, and hence form a basis for $\mathbb{Z}_3^3$ over $\mathbb{Z}_3$. Fix a representation for $\mathbb{Z}_3^3$ which comprises of representing all elements of $\mathbb{Z}_3^3$ as the set of coordinate vectors with respect to the ordered bases $\{s_1, s_2, s_3\}$. This means that $s_1 = (1, 0, 0)$, $s_2 = (0, 1, 0)$ and $s_3 = (0, 0, 1)$. Since S does not contain a non-empty zero-sum subsequence of length at most 3, we find that

$$s_4, s_5 \notin \{(0,0,0),(0,0,2),(0,2,0),(0,2,2),(2,0,0),(2,2,0),(2,0,2)\}.$$

Since we are supposing that there does not exist a subsequence $s_i s_j s_k$ of S such that $s_i + s_j = s_k$, we find that s_4 and s_5 cannot belong to the set

$$\{(0,1,1),(0,2,1),(0,1,2),(1,1,0),(1,2,0),(1,0,1),(1,0,2),(2,1,0),(2,0,1)\}.$$

Hence $s_4, s_5 \in A \cup B \cup C \cup D$ where

$$\begin{aligned} A &= \{(1,1,1),(1,2,2)\}, & B &= \{(1,1,2),(1,2,1)\}, \\ C &= \{(2,1,2),(2,2,1)\}, & D &= \{(2,1,1),(2,2,2)\}. \end{aligned}$$

We aim to prove the following claims for distinct $i, j \in \{4,5\}$:

1. Each set A, B, C, D contains at most one out of s_i and s_j.
2. If $s_i \in A$ then $s_j \notin B \cup C \cup D$.
3. If $s_i \in B$ then $s_j \notin C \cup D$.
4. If $s_i \in C$ then $s_j \notin D$.

Before we prove these claims, let us show how we can use them to obtain a contradiction. If $s_4 \in A$ then (1) and (2) imply a contradiction to $s_5 \in A \cup B \cup C \cup D$. Suppose $s_4 \in B$. Then (1) and (3) imply $s_5 \notin B \cup C \cup D$. Now note that (2) implies $s_5 \notin A$ in order to obtain a contradiction. Similarly, if $s_4 \in C$ then (1) and (4) imply $s_5 \notin C \cup D$, and (2) and (3) imply $s_5 \notin A \cup B$ which again gives a contradiction. Now suppose $s_4 \in D$. Then we obtain a contradiction by noting that (2), (3), (4), and (1) respectively imply that s_5 cannot be in A, B, C, or D.

It remains to prove the four claims above in order to complete the proof. We prove (1) first. Noting that s_4 and s_5 are distinct and S does not contain a zero-sum subsequence of length 3, we find that s_4 and s_5 cannot both be in A and cannot both be in B. Since we are assuming the assertion in the statement of the lemma does not hold, we find that s_4 and s_5 cannot both be in C and cannot both be in D. This completes the proof of (1). Fix $i \neq j$ in $\{4, 5\}$. Noting that

$$(1,1,1) + (0,0,1) = (1,1,2), \qquad (1,1,1) + (0,1,0) = (1,2,1),$$
$$(1,1,2) + (0,1,0) = (1,2,2), \qquad (1,2,1) + (0,0,1) = (1,2,2),$$

we find that if $s_i \in A$ then $s_j \notin B$. Noting that

$$(1,1,1) + (2,1,2) + (0,1,0) = (0,0,0),$$
$$(1,1,1) + (2,2,1) + (0,0,1) = (0,0,0),$$
$$(1,2,2) + (2,1,2) = (0,0,1),$$
$$(1,2,2) + (2,2,1) = (0,1,0),$$

we find that if $s_i \in A$ then $s_j \notin C$. Noting that

$$(1,1,1) + (1,0,0) = (2,1,1), \qquad (1,1,1) + (2,2,2) = (0,0,0),$$
$$(1,2,2) + (2,1,1) = (0,0,0), \qquad (1,2,2) + (1,0,0) = (2,2,2),$$

we find that if $s_i \in A$ then $s_j \notin D$. This completes the proof of (2). Using similar calculations we find that if $s_i \in B$ then $s_j \notin C \cup D$ and if $s_i \in C$ then $s_j \notin D$. So (3) and (4) follow and the lemma is proved. □

Lemma 4.3.2 (Lemma 1 (3) in [3])**.** *Every sequence of* 9 *pairwise distinct elements of* $\mathbb{Z}_3^3$ *contains a non-empty zero-sum subsequence of length at most* 3.

Proof. We prove this result by creating a computer program which generates all sequences over $\mathbb{Z}_3^3$ of length 9 consisting of nine pairwise distinct elements and no non-empty zero-sum subsequence of length at most 3. We shall refer to this computer program as *CPT9* and describe it in Section 8.1.1. We find that CPT9 does not generate a counterexample to the statement of the lemma which completes the proof. □

The following result allows us to avoid using Theorem 2 in [3] (the proof of which is over 6 pages long) in the proof of the equality $D(G) = 1 + d^*(G)$ for $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$.

Lemma 4.3.3. *Let G and T be finite abelian groups with $T \cong \mathbb{Z}_3^3$. Let S be a sequence over G containing a subsequence Q of length* 10*. Let $\phi : G \longrightarrow T$ be a map satisfying the following properties:*

- *There do not exist two disjoint T-zero-sum subsequences in Q with respect to ϕ.*
- *Every T-zero-sum subsequence of Q with respect to ϕ has the same value equal to some fixed element $c \in G$.*

Then S is not zero-sum free.

Proof. Note that Q does not contain a T-zero-sum subsequence of length $l \in \{1, 2, 3, 8, 9, 10\}$. Indeed, if Q contains a T-zero-sum subsequence of length at most 3 then we can remove it from Q to obtain a sequence of length at least 7. Since $D(\mathbb{Z}_3^3) = 7$, this sequence of 7 elements contains a T-zero-sum subsequence. Hence we obtain two disjoint T-zero-sum subsequences in Q which contradicts our assumption. If Q contains a T-zero-sum subsequence U of length $l \in \{8, 9, 10\}$ then, since $D(\mathbb{Z}_3^3) = 7$, the sequence U is the union of two disjoint T-zero-sum subsequences of Q which again contradicts our assumption.

Let $Q = q_1 \cdots q_{10}$. Since 9 pairwise distinct elements in $\mathbb{Z}_3^3$ contain a non-empty zero-sum subsequence of length at most 3 (see Lemma 4.3.2), we have $\phi(Q)$ contains at most 8 pairwise distinct elements. Hence Q contains a subsequence $q_i q_j q_m q_n$ such that $\phi(q_i) = \phi(q_j)$ and $\phi(q_m) = \phi(q_n)$. Now view $\mathbb{Z}_3^3$ as a 3-dimensional vector space over $\mathbb{Z}_3$ and note that $\phi(q_i)$ and $\phi(q_m)$ are linearly independent. We claim that there exists an element q_k in Q such that

$$\phi(q_k) \notin \mathrm{Span}_{\mathbb{Z}_3}\{\phi(q_i), \phi(q_m)\}.$$

Suppose for a contradiction that $\phi(q_k) \in \mathrm{Span}_{\mathbb{Z}_3}\{\phi(q_i), \phi(q_m)\}$ for all elements q_k in Q. Then, we can consider $\phi(Q)$ as a sequence over $\langle \phi(q_i), \phi(q_m) \rangle$. Now since $\langle \phi(q_i), \phi(q_m) \rangle \cong \mathbb{Z}_3^2$ and $D(\mathbb{Z}_3^2) = 5$ and Q is a sequence of length 10, we can deduce that Q contains two disjoint T-zero-sum subsequences. This contradicts our assumption. Without loss of generality suppose $i = 1, j = 2, m = 3, n = 4, k = 5$. Now fix a representation for $\mathbb{Z}_3^3$ which comprises of representing all elements of $\mathbb{Z}_3^3$ as the set of coordinate vectors with respect to the ordered basis $\{\phi(q_1), \phi(q_3), \phi(q_5)\}$. This means that $\phi(q_1) = \phi(q_2) = (1, 0, 0), \phi(q_3) = \phi(q_4) = (0, 1, 0)$, and $\phi(q_5) = (0, 0, 1)$.

We now create a computer program, which we call *CPT10*, to generate all sequences over $\mathbb{Z}_3^3$ of length 10 which do not contain a zero-sum subsequence of length $l \in \{1, 2, 3, 8, 9, 10\}$ and which contain the subsequence

$$(1, 0, 0)(1, 0, 0)(0, 1, 0)(0, 1, 0)(0, 0, 1).$$

We describe CPT10 in Section 8.1.2. We find that CPT10 generates 1173 sequences. Let $X = x_1 \cdots x_{10}$ be an arbitrary sequence from the 1173 sequences generated by CPT10. We present a method which shows that if $X = \phi(Q)$ then S is not zero-sum free. After relabelling if necessary, fix $\phi(q_1) = x_1, \ldots, \phi(q_{10}) = x_{10}$. Now compute all T-zero-sum subsequences of Q. Since every T-zero-sum subsequence of Q has the same value equal to some fixed element c, each T-zero-sum subsequence gives us a linear homogenous equation in the 11 variables $q_1, \ldots, q_{10}$ and c. Hence we obtain a homogenous linear system of simultaneous equations in the 11 variables $q_1, \ldots, q_{10}$ and c. The idea now is to reduce this system of equations to a simpler one. To do this we put the coefficients of the 11 variables arising from these T-zero-sum subsequences as the rows of a matrix A_X with 11 columns and find its row Hermite normal form. We automate this process to find the row Hermite normal form of A_X for all sequences X in a computer program which we call *CPT10CNTR*. This program is described in Section 8.1.3. We find that the row Hermite normal form of A_X for each sequence X contains the row

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This implies $c = 0_G$ and hence S is not zero-sum free in all 1173 possibilities for $\phi(Q)$. □

Remark 4.3.4. Computing the row Hermite normal form of a matrix A is equivalent to performing a series of *elementary unimodular row operations* on A consisting of:

- Interchanging two rows of A.
- Multiplying a row of A by -1.
- Adding an integer multiple of one row of A to another.

This can be deduced by combining the row analogues of Theorem 2.4.3 and Algorithm 2.4.4 in [7].

Definition 4.3.5. Let $S = s_1 \cdots s_n$ be a sequence over a group G. Given $g \in G$, the *multiplicity* of g in S is defined to be the number of times g occurs in the multiset $\{s_1, \ldots, s_n\}$.

Lemma 4.3.6. *(Proposition* 4 *in [3]). Every sequence over* $\mathbb{Z}_3^3$ *of length* 10 *contains a non-empty zero-sum subsequence of length at most* 4.

Proof. Let $S = s_1 \cdots s_{10}$ be a sequence over $\mathbb{Z}_3^3$ of length 10 and suppose for a contradiction that S does not contain a non-empty zero-sum subsequence of length at most 4. If S does not contain five pairwise distinct elements then S contains an element with multiplicity at least 3 and hence a zero-sum subsequence of length 3. So S must contain (at least) five pairwise distinct elements. Consequently, Lemma 4.3.1 tells us that S contains a sequence $s_i s_j s_k$ of three pairwise distinct elements such that $s_i + s_j = s_k$. Without loss of generality assume $i = 1$, $j = 2$ and $k = 3$. Viewing $\mathbb{Z}_3^3$ as a vector space over $\mathbb{Z}_3$, we claim that s_1 and s_2 are linearly independent. Indeed this is the case else $s_1 = s_2$ or $s_1 s_2$ contains a zero-sum subsequence of length at most 2. Now pick an element $t \in \mathbb{Z}_3^3$ such that $t \notin \mathrm{Span}_{\mathbb{Z}_3}\{s_1, s_2\}$ and fix a representation for $\mathbb{Z}_3^3$ which comprises of representing all elements of $\mathbb{Z}_3^3$ as the set of coordinate vectors with respect to the ordered basis $\{t, s_1, s_2\}$. Hence, we have

$$s_1 = (0, 1, 0), s_2 = (0, 0, 1), s_3 = (0, 1, 1).$$

The next step of the proof is to create a computer program which generates all sequences over $\mathbb{Z}_3^3$ of length 10 containing $(0, 1, 0)(0, 0, 1)(0, 1, 1)$ as a subsequence and no non-empty zero-sum subsequence of length at most 4. We shall refer to this program as *CPT10F* and describe it in Section 8.1.4. Using CPT10F we find there does not exist a sequence over $\mathbb{Z}_3^3$ of length 10 containing $(0, 1, 0)(0, 0, 1)(0, 1, 1)$ as a subsequence and no non-empty zero-sum subsequence of length at most 4. This completes the proof. □

Corollary 4.3.7 (Proposition 5 in [3]). *Every sequence over* $\mathbb{Z}_3^3$ *of length* 11 *contains* 2 *non-empty disjoint zero-sum subsequences.*

Proof (derived from the proof of Proposition 5 *in [3]).* Let S be a sequence over $\mathbb{Z}_3^3$ of length 11. By Lemma 4.3.6, we have that S contains a non-empty zero-sum subsequence S_1 of length at most 4. Remove S_1 from S to obtain a subsequence of S of length at least 7. Since $D(\mathbb{Z}_3^3) = 7$, this subsequence contains a non-empty zero-sum subsequence, say S_2. It remains to note that S_1 and S_2 are two non-empty disjoint zero-sum subsequences of S. □

The authors of [3] refer to using part (i) of the proof of Theorem 2 in [3] in their proof of the equality $D(G) = 1 + d^*(G)$ for $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ (see the 3rd line from the bottom of page 19 in [3]). However, we cannot see how the assumptions needed to use part (i) of the proof of Theorem 2 are satisfied at the place where the authors of [3] claim it can be applied. The following result allows us to avoid using part (i) of the proof of Theorem 2 in the proof of the equality $D(G) = 1 + d^*(G)$ for $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$.

Lemma 4.3.8. *Let G and T be finite abelian groups with $T \cong \mathbb{Z}_3^3$. Let S be sequence over G containing a subsequence Q of length 13. Let $\phi : G \longrightarrow T$ be a map satisfying the following properties:*

- *There does not exist a T-zero-sum subsequence in Q with respect to ϕ of length at most 3.*
- *Every two disjoint T-zero-sum subsequences of Q with respect to ϕ have the same value equal to some fixed element $c \in G$.*

Then S is not zero-sum free.

Proof. If $\phi(Q)$ contains 9 pairwise distinct elements then Lemma 4.3.2 implies Q contains a T-zero-sum subsequence of length at most 3. This contradicts our assumption. So $\phi(Q)$ contains at most 8 pairwise distinct elements. This means that we can assume $Q = q_1 \cdots q_{13}$ with

$$\phi(q_i) = \phi(q_{i+1})$$

and $\phi(q_i)$ pairwise distinct for all $i \in \{1, 3, 5, 7, 9\}$. Using Lemma 4.3.1 we deduce that there exist pairwise distinct $m, n, p \in \{1, 3, 5, 7, 9\}$ such that

$$\phi(q_m) + \phi(q_n) = \phi(q_p).$$

Now view $\mathbb{Z}_3^3$ as a 3-dimensional vector space over $\mathbb{Z}_3$ and note that $\phi(q_m)$ and $\phi(q_n)$ are linearly independent. Pick $t \in \mathbb{Z}_3^3$ with $t \notin \mathrm{Span}_{\mathbb{Z}_3}\{\phi(q_m), \phi(q_n)\}$

and form the ordered basis $\{t, \phi(q_m), \phi(q_n)\}$ of $\mathbb{Z}_3^3$. Now fix a representation for $\mathbb{Z}_3^3$ which comprises of representing all elements of $\mathbb{Z}_3^3$ as the set of coordinate vectors with respect to this basis. This means that $\phi(q_m) = (0, 1, 0)$, $\phi(q_n) = (0, 0, 1)$, $\phi(q_p) = (0, 1, 1)$, and $t = (1, 0, 0)$. Without loss of generality suppose $m = 1$, $n = 3$ and $p = 5$. We claim that

$$\phi(q_7) \notin \mathrm{Span}_{\mathbb{Z}_3}\{\phi(q_1), \phi(q_3)\}.$$

Indeed, if $\phi(q_7) \in \mathrm{Span}_{\mathbb{Z}_3}\{\phi(q_1), \phi(q_3)\}$ then it follows that $\phi(q_7) = (0, a, b)$ for some $a, b \in \{0, 1, 2\}$. Since $\phi(q_i)$ are pairwise distinct for all $i \in \{1, 3, 5, 7\}$, we deduce that

$$\phi(q_7) \in \{(0, 0, 0)(0, 2, 0)(0, 2, 1)(0, 2, 2)(0, 1, 2)(0, 0, 2)\}.$$

If $\phi(q_7) = (0, 0, 0)$ then Q contains a T-zero-sum subsequence of length 1. If $\phi(q_7) \in \{(0, 2, 0), (0, 2, 2)(0, 0, 2)\}$ then Q contains a T-zero-sum subsequence of length 2. If $\phi(q_7) \in \{(0, 2, 1), (0, 1, 2)\}$ then Q contains a T-zero-sum subsequence of length 3. Each of the last three statements contradict our assumption. This proves the claim. This means we can pick $t = \phi(q_7)$.

We now create a computer program, which we call *CPT13*, to generate all sequences over $\mathbb{Z}_3^3$ of length 13 without a non-empty zero-sum subsequence of length at most 3 containing the subsequence

$$(0, 1, 0)(0, 1, 0)(0, 0, 1)(0, 0, 1)(0, 1, 1)(0, 1, 1)(1, 0, 0)(1, 0, 0).$$

We describe CPT13 in Section 8.1.5. We find that CPT13 generates 149 sequences. Let $X = x_1 \cdots x_{13}$ be an arbitrary sequence from the 149 sequences generated by CPT13. Using the same method as in the proof of Lemma 4.3.3 we show that if $X = \phi(Q)$ then S is not zero-sum free. After relabelling if necessary, fix $\phi(q_1) = x_1, \ldots, \phi(q_{13}) = x_{13}$. Note that every T-zero-sum subsequence of Q of length at most 6 has the same value equal to c. Indeed, if U is a T-zero-sum subsequence of Q of length at most 6 then remove U from Q to obtain a sequence of at least 7 elements. Since $D(\mathbb{Z}_3^3) = 7$, these 7 elements contain a T-zero-sum subsequence, say U'. Now U and U' are two disjoint T-zero-sum subsequences of Q and hence have the same value equal to c by assumption. Now compute all T-zero-sum subsequences of Q of length at most 6. Since each such T-zero-sum subsequence of Q has the same value equal to some fixed element c, each such T-zero-sum subsequence gives us a linear homogenous equation in the 14 variables $q_1, \ldots, q_{13}$ and c. Hence we obtain a homogenous linear system of simultaneous equations in the 14 variables $q_1, \ldots, q_{13}$ and c. The idea now is to reduce this system of equations to a simpler one. To do this we put the coefficients of the 14 variables arising from these T-zero-sum subsequences as the rows of a matrix A_X with 14 columns and find its row Hermite normal form. We automate this process to find the row Hermite normal form of A_X for all sequences X in a computer program which we call *CPT13CNTR*. This program is described in Section 8.1.6. We find that the row Hermite normal form of A_X for each sequence X contains the row

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This implies $c = 0_G$ and hence S is not zero-sum free in all 149 possibilities for $\phi(Q)$. $\square$

Corollary 4.3.9 (Proposition 6 in [3]). *Every sequence over* $\mathbb{Z}_3^3$ *of length* 15 *contains* 3 *non-empty disjoint zero-sum subsequences.*

Proof (derived from the proof of Proposition 6 *in [3]).* Let S be a sequence over $\mathbb{Z}_3^3$ of length 15. By Lemma 4.3.6, we have that S contains a non-empty zero-sum subsequence S_1 of length at most 4. Remove S_1 from S to obtain a subsequence of S of length at least 11. By Corollary 4.3.7 this subsequence contains 2 non-empty disjoint zero-sum subsequences, say S_2 and S_3. It remains to note that S_1, S_2 and S_3 are three non-empty disjoint zero-sum subsequences of S. $\square$

Lemma 4.3.10. *Let S be a sequence of 16 non-zero elements of $\mathbb{Z}_3^3$ containing no zero-sum subsequence of length 3 and no pair of disjoint zero-sum subsequences of length 2. Then S is zero-sum.*

Proof. Firstly we note the following: any subsequence of S of length 9 contains 5 pairwise distinct elements else S contains an element with multiplicity at least 3 which contradicts the assumption that S does not contain a zero-sum subsequence of length 3.

We now make a sequence of claims to determine four elements of S. Let $S = s_1 \cdots s_{16}$. We claim that S contains a subsequence $s_i s_j s_k$ of pairwise distinct elements such that $s_i + s_j = s_k$. We split the proof of this claim in two cases. Suppose S does not contain a zero-sum subsequence of length 2. In this case S does not contain a non-empty zero-sum subsequence of length at most 3. Applying Lemma 4.3.1 to 5 pairwise distinct elements in S allows us to prove the claim in this case. Now suppose S contains a zero-sum subsequence of length 2. Removing this zero-sum subsequence of length 2 from S leaves us with a sequence of length 14 containing no non-empty zero-sum subsequence of length at most 3. Similar to the previous case, applying Lemma 4.3.1 proves the claim in this case. Without loss of generality, let $i = 1$, $j = 2$ and $k = 3$. Viewing $\mathbb{Z}_3^3$ as a vector space over $\mathbb{Z}_3$, we now claim that s_1 and s_2 are linearly independent over $\mathbb{Z}_3^3$. Indeed this is the case, otherwise we obtain a contradiction to the fact that s_1 and s_2 are distinct or the assumption that S consists entirely of non-zero elements. Next we claim that there exists an element s_m in S such that $s_m \notin \mathrm{Span}_{\mathbb{Z}_3}\{s_1, s_2\}$. In order to prove this claim suppose for a contradiction that $s_m \in \mathrm{Span}_{\mathbb{Z}_3}\{s_1, s_2\}$ for all elements s_m of S. Note that for any element s_m of S we have

$$s_m \notin \{(0,0,0), s_1 + 2s_2\}$$

else S contains a zero-sum subsequence of length 1 or 3. Hence S contains at most $9 - 2 = 7$ pairwise distinct elements. However this implies that S contains an element with multiplicity at least 3 which contradicts the assumption that S does not contain a zero-sum subsequence of length 3. This proves the claim. Without loss of generality suppose $s_4 \notin \mathrm{Span}_{\mathbb{Z}_3}\{s_1, s_2\}$. This means that we can represent elements of $\mathbb{Z}_3^3$ as the set of coordinate vectors with respect to the ordered basis $\{s_4, s_1, s_2\}$. Hence, we have that $s_1 = (0, 1, 0)$, $s_2 = (0, 0, 1)$, $s_3 = (0, 1, 1)$, and $s_4 = (1, 0, 0)$.

The next step of the proof is to create a computer program which cuts down on the number of possibilities for S given that we have determined $s_1 \cdots s_4$ as above. We shall refer to this program as *CPT16*. In simple terms, CPT16 generates all sequences of 16 non-zero elements over $\mathbb{Z}_3^3$ containing the subsequence

$$(0, 1, 0)(0, 0, 1)(0, 1, 1)(1, 0, 0),$$

which contain no zero-sum subsequence of length 3 and no subsequence from a list of 19 sequences of length 4 over $\mathbb{Z}_3^3$ made up of the union of a pair of zero-sum sequences of length 2. We describe CPT16 in detail in Section 8.1.7 and only present the use of its output here. We find that CPT16 generates five sequences of length 16 over $\mathbb{Z}_3^3$ each consisting of 8 pairwise distinct elements repeated twice; the sequences of 8 pairwise distinct elements are the following:

1. (0,1,0) (0,0,1) (0,1,1) (1,0,0) (1,1,0) (1,2,1) (2,2,1) (1,2,2),
2. (0,1,0) (0,0,1) (0,1,1) (1,0,0) (2,1,0) (2,0,1) (2,2,1) (2,1,2),
3. (0,1,0) (0,0,1) (0,1,1) (1,0,0) (2,1,0) (1,2,1) (1,1,1) (1,0,2),
4. (0,1,0) (0,0,1) (0,1,1) (1,0,0) (1,2,0) (2,0,1) (1,1,1) (1,1,2)
5. (0,1,0) (0,0,1) (0,1,1) (1,0,0) (1,0,1) (2,1,2) (1,2,2) (1,1,2).

We deduce that the value of each of the five sequences generated by CPT16 is $(0, 0, 0)$ which completes the proof. As a passing remark, it is not hard to show that the five sequences generated by CPT16 do not contain a pair of disjoint zero-sum subsequences of length 2, which implies that those five sequences are precisely the only possibilities for S. $\square$

Remark 4.3.11. The 10th line from the bottom of page 19 in [3] states that every sequence of length 16 over $\mathbb{Z}_3^3$ without a zero-sum subsequence of length 3 is zero-sum. However, we find a counterexample to this statement as follows. Define S to be the sequence over $\mathbb{Z}_3^3$ consisting of the following 8 elements each repeated twice:

$$(0,1,0),(0,0,1),(0,1,1),(1,0,0),(1,1,1),(2,1,0),(2,0,1),(2,0,0).$$

Implementing the pseudocode of the function described in Figure 8.2 in the computer algebra system Magma, one can easily check that S does not contain a zero-sum subsequence of length 3. It turns out that $|S| = (1,2,2)$.

The following lemma originates from [19]. Note that this article is not easily available online: for a statement of the following result in an easily available piece of published literature see Theorem 1.1 in [10].

Lemma 4.3.12 ([19]). *Every sequence over $\mathbb{Z}_3^3$ of length 19 contains a zero-sum subsequence of length 3.*

4.4 The equality $D(G) = 1 + d^*(G)$

We present a proof of the equality $D(G) = 1 + d^*(G)$ for $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ in this section.

Theorem 4.4.1 (Theorem 1 in [3]). *Let $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ where d is an arbitrary positive integer. Then $D(G) = 1 + d^*(G)$.*

Proof (derived from the proof of Theorem 1 in [3]). If $d \in \{1,3\}$ then G is a 3-group and therefore the result is proved by Theorem 2.1.1. If $d = 2$ then

$G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6$ and the result is proved by Theorem 2.1.3. So we can assume $d \geq 4$.

Let S be an arbitrary sequence of length $1 + d^*(G) = 3d + 4$ over G. We claim that S is not zero-sum free. We make the following observation in order to prove this claim. Define

$$H := \{(0,0,z) \in G \mid z \equiv 0 \bmod 3\} < G.$$

Then $H \cong \mathbb{Z}_d$. Define $T := G/H \cong \mathbb{Z}_3^3$. Since $D(H) = d$, by Lemma 4.2.3 it is sufficient to find d disjoint T-zero-sum subsequences in S with respect to the canonical homomorphism $\phi : G \longrightarrow T$ in order to prove the claim.

We start by searching for T-zero-sum subsequences of length 3 in S as follows. Suppose there exists a T-zero-sum subsequence S' in S with length 3. Remove S' from S to form a subsequence Q_1 of S. Now repeat the search on Q_1 to form Q_2 and so on until what remains is a subsequence Q of S without any T-zero-sum subsequences of length 3. Note that since the length of S is $3d + 4$, the length of each Q_i and hence Q, is of the form $3k + 1$ for some $k \in \mathbb{N} \cup \{0\}$. Now since any subsequence of S of length 19 contains a T-zero-sum subsequence of length 3 (see Lemma 4.3.12), we have that the length of Q is at most 18. That is, Q is a sequence of length $3k+1$ for some $0 \leq k \leq 5$. We compute the number of disjoint T-zero-sum subsequences of length 3 removed from S to form Q to be

$$\frac{(3d+4) - (3k+1)}{3} = d - k + 1.$$

So we have that S contains $d-k+1$ disjoint T-zero-sum subsequences which are not in Q (note that $d - k + 1 \geq 0$ since $d \geq 4$). Now we deal with the cases $k = 0, 1, 2, 3, 4, 5$ separately.

Case (i): Suppose $k \in \{0, 1\}$. Then $d-k+1 \geq d$ and hence S contains at least d disjoint T-zero-sum subsequences. The proof is complete in this case.

Case (ii): Suppose $k = 2$. Then S contains $d - 1$ disjoint T-zero-sum subsequences not in Q. However note that the length of Q in this case is 7. Therefore, since $D(\mathbb{Z}_3^3) = 7$, we deduce that Q contains a T-zero-sum subsequence. So S contains d disjoint T-zero-sum subsequences and the proof is complete in this case.

Case (iii): Suppose $k = 5$. In this case Q is a sequence of length 16 and S contains $d-4$ disjoint T-zero-sum subsequences not in Q. If Q contains a T-zero-sum subsequence of length 1 then removing it from Q leaves us with a sequence of length 15 which by Corollary 4.3.9 contains 3 disjoint T-zero-sum subsequences. Hence we can obtain 4 disjoint T-zero-sum subsequences in Q in this instance. If Q contains a pair of disjoint T-zero-sum subsequences of length 2 then we can remove them from Q to obtain a sequence of length 12 which, by Corollary 4.3.7, contains 2 disjoint T-zero-sum subsequences. Hence we can obtain 4 disjoint T-zero-sum subsequences in Q in this instance too. Now suppose Q does not contain a T-zero-sum subsequence of length 1 and contains no pair of disjoint T-zero-sum subsequences of length 2. Then Lemma 4.3.10 implies Q is a T-zero-sum sequence. Hence Corollary 4.3.9 implies Q is the union of 4 disjoint T-zero-sum subsequences. So we conclude that S contains d disjoint T-zero-sum subsequences in all scenarios which completes the proof in this case.

Case (iv): Suppose $k = 3$. In this case Q is a sequence of length 10 and S contains $d-2$ disjoint T-zero-sum subsequences $S_1, \ldots, S_{d-2}$ not in Q. Suppose for a contradiction that S is zero-sum free. Then we claim that every T-zero-sum subsequence of Q has the same value equal to $|S_{d-2}|$. Indeed, if S_{d-1} is a T-zero-sum subsequence of Q then S contains $d-1$ disjoint T-zero-sum subsequences. It remains to apply Corollary 4.2.5 whilst noting that $d-2 \geq 1$ in order to prove the claim. If Q contains two disjoint T-zero-sum subsequences then S contains d disjoint T-zero-sum subsequences which contradicts the assumption that S is zero-sum free. If Q does not contain two disjoint T-zero-sum subsequences then we can use Lemma 4.3.3 to deduce that S is not zero-sum free and hence obtain a contradiction. This completes the proof in this case.

Case (v): Suppose $k = 4$. In this case Q is a sequence of length 13 and S contains $d-3$ disjoint T-zero-sum subsequences $S_1, \dots, S_{d-3}$ not in Q. Suppose Q contains a T-zero-sum subsequence U of length at most 2. In this case, remove U from Q to obtain a sequence U' of length at least 11. By Corollary 4.3.7, the sequence U' contains 2 disjoint T-zero-sum subsequences, and hence we obtain 3 disjoint T-zero-sum subsequences in Q. Thus S contains d disjoint T-zero-sum subsequences and the proof is complete in this case. Now suppose that Q does not contain a T-zero-sum subsequence of length at most 3. Suppose for a contradiction that S is zero-sum free. Then every two disjoint T-zero-sum subsequences of Q have the same value equal to $|S_{d-3}|$. To see this, use Corollary 4.2.5 whilst noting that $d-3 \geq 1$ and S contains $d-3$ disjoint T-zero-sum subsequences not in Q. This means that we can use Lemma 4.3.8 to deduce that S is not zero-sum free. This is a contradiction. The proof is complete. $\square$

Chapter 5

Some results on sequences over $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{2p}$

5.1 Motivation

Let $p > 2$ be an arbitrary prime number. In this chapter we present three results about sequences over the group $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{2p}$. After a preliminaries section (Section 5.2), the first result we present (Section 5.3) provides sufficient conditions for sequences over $\mathbb{Z}_p^3 \oplus \mathbb{Z}_2$ of length $4p - 2$ to contain a non-empty zero-sum subsequence. Viewing elements of $\mathbb{Z}_p^3 \oplus \mathbb{Z}_2$ as 2-tuples, the second and third results (Section 5.4) allow us to determine some of the first components of elements of zero-sum free sequences over $\mathbb{Z}_p^3 \oplus \mathbb{Z}_2$ of length $4p - 2$ up to isomorphism.

The motivation for the results in this chapter stems from the problem of finding the Davenport constant of the group $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{2p}$ for an arbitrary prime number $p > 2$. The value of the Davenport constant of $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{2p}$ is conjectured to be $1 + d^*(G) = 4p - 2$ from Conjecture 1.2.3. It is therefore natural to investigate sufficient conditions for sequences over $\mathbb{Z}_p^3 \oplus \mathbb{Z}_2$ of length $4p - 2$ to contain a non-empty zero-sum subsequence as well as the structure of sequences over $\mathbb{Z}_p^3 \oplus \mathbb{Z}_2$ which may give rise to a possible counterexample to Conjecture 1.2.3. The results in this chapter are not enough to establish the conjecture for this class of groups. However, the results will be used in Chapter 6 to establish the conjecture when $p = 5$. Moreover, we hope these results will contribute to a general result in future.

5.2 Preparatory material

In this section we present the results that are used in Section 5.3 and Section 5.4. The reader may notice that the results in this section generalise some of the ideas used in the proof of the equality $D(G) = 1 + d^*(G)$ for $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ in Chapter 4.

Definition 5.2.1. The *set of subsums* $[S]$ of a sequence S over a group G is defined to be the subset of G consisting of the values of all non-empty subsequences of S.

Proposition 5.2.2 (Proposition 5.1.4 (1) in [14])**.** *Let S be a zero-sum free sequence over a finite abelian group G of length $D(G)-1$. Then $[S] = G \setminus \{0_G\}$.*

Proof. As S is zero-sum free, we have that $[S] \subseteq G \setminus \{0_G\}$. So it remains to show $G \setminus \{0_G\} \subseteq [S]$. Pick $g \in G \setminus \{0_G\}$ and form the sequence

$$T := S \cup -g$$

over G. Since the length of T is $D(G)$, we deduce that it contains a non-empty zero-sum subsequence U. Since S is zero-sum free and $g \neq 0_G$, the sequence U must be of the form

$$U = V \cup -g$$

for some non-empty subsequence V of S. We have that

$$0_G = |U| = |V| - g$$

which implies that $g = |V|$. Now note that $|V| \in [S]$ as V is a non-empty subsequence of S. This means that $g \in [S]$. The proof is complete. $\square$

Definition 5.2.3. Let G and H be finite abelian groups and let

$$S = (x_1, y_1) \cdots (x_l, y_l)$$

be a sequence over $G \oplus H$. Given a non-empty subsequence $T := x_{i_1} \cdots x_{i_t}$ of $x_1 \cdots x_l$, we define the *extension of T into S*, denoted T^S, to be the subsequence

$$(x_{i_1}, y_{i_1}) \cdots (x_{i_t}, y_{i_t})$$

of S. If T is the empty sequence then define T^S to be the empty sequence as well.

Lemma 5.2.4. *Let H be a finite abelian group and define $G := H \oplus \mathbb{Z}_2$. Let*

$$S := (x_1, y_1) \cdots (x_l, y_l)$$

be a sequence over G. If either of the following hold then S is not zero-sum free:

(i) The sequence $x_1 \cdots x_l$ contains two non-empty disjoint zero-sum subsequences;

(ii) The sequence $x_1 \cdots x_l$ contains a non-empty zero-sum subsequence of even length and $y_1 = \cdots = y_l$.

Proof. Suppose $x_1 \cdots x_l$ contains two non-empty disjoint zero-sum subsequences $T_1 := x_{i_1} \cdots x_{i_v}$ and $T_2 := x_{j_1} \cdots x_{j_w}$. If $|y_{i_1} \cdots y_{i_v}| = 0$ then T_1^S is a non-empty zero-sum subsequence of S. If $|y_{j_1} \cdots y_{j_w}| = 0$ then T_2^S is a non-empty zero-sum subsequence of S. If $|y_{i_1} \cdots y_{i_v}| = |y_{j_1} \cdots y_{j_w}| = 1$ then $(T_1 \cup T_2)^S$ is a non-empty zero-sum subsequence of S. Hence in all cases S is not zero-sum free.

Suppose $T := x_{i_1} \cdots x_{i_j}$ is a non-empty zero-sum subsequence of $x_1 \cdots x_l$ of even length j and $y_1 = \cdots = y_l$. We claim that T^S is a non-empty zero-sum subsequence of S, and hence S is not zero-sum free. If $y_1 = \cdots = y_l = 0$ then the claim is obvious. Suppose $y_1 = \cdots = y_l = 1$. Then the value of the sequence T^S is $(0_H, j)$. Since j is even, we have that $j \equiv 0 \pmod 2$. Hence T^S is zero-sum. □

Lemma 5.2.5. *Let S be a sequence over a finite abelian group G and suppose S contains a non-empty zero-sum subsequence T such that either*

(i) there exists a positive integer e such that the length of T is at most e and the length of S is at least $e + D(G)$, or

(ii) the length of T is strictly greater than $D(G)$.

Then S contains two non-empty disjoint zero-sum subsequences.

Proof. Suppose the length of S is at least $e + D(G)$ and the length of T is at most e for some fixed positive integer e. Then removing T from S we obtain a sequence U over G of length at least $D(G)$. We deduce that U contains a non-empty zero-sum subsequence V. It remains to note that T and V are two non-empty disjoint zero-sum subsequences of S in order to complete the proof of this case.

Suppose the length of T is strictly greater than $D(G)$. Then T contains a non-empty zero-sum subsequence T_1 of length at most $D(G)$. Since T is zero-sum, the sequence T_2 obtained by removing T_1 from T is a non-empty sequence which is also zero-sum. Hence T_1 and T_2 are two non-empty disjoint zero-sum subsequences of S. This completes the proof. □

Definition 5.2.6. A subsequence T of a sequence S over a group is called *proper* if T is not the empty sequence and the length of T is strictly smaller than the length of S.

Lemma 5.2.7. *Let H be a finite abelian group and define $G := H \oplus \mathbb{Z}_2$. Let*

$$S = (x_1, y_1) \cdots (x_n, y_n)$$

be a sequence over G such that $y_1 = \cdots = y_r = 1$ and $y_{r+1} = \cdots = y_n = 0$ for some odd $r \geq 1$. Define $t := -(x_1 + \cdots + x_n)$. If the sequence $S \cup (t, 1)$ over G contains a proper zero-sum subsequence then S is not zero-sum free.

Proof. Let T be a proper zero-sum subsequence of $S \cup (t, 1)$ and suppose for a contradiction that S is zero-sum free. Then T must be of the form

$$T = U \cup (t, 1)$$

for some proper subsequence U of S. Now note that since r is odd we have

$$|S \cup (t, 1)| = (x_1 + \cdots + x_n + t, r + 1) = (0_H, 0) = |T|.$$

Hence

$$|S| = |U|.$$

Consequently, since U is a proper zero-sum subsequence of S, we find that the subsequence of S formed by removing U from S is a non-empty zero-sum sequence. This contradicts the assumption that S is zero-sum free and hence proves the result. $\square$

The following theorem is a consequence of results proved by Weidong Gao, Alfred Geroldinger and Christian Reiher (see Remark 5.2.9).

Theorem 5.2.8. *Fix a prime number p and let S be a sequence of length $3p - 3$ over a group G such that $G \cong \mathbb{Z}_p^2$. If S does not contain a non-empty zero-sum subsequence of length at most p then S must be of the form*

$$S = a \cdots ab \cdots bc \cdots c$$

for some pairwise distinct elements $a, b, c \in G$ each having multiplicity $p-1$ in S.

Remark 5.2.9. We are unable to find an explicit body of text referring to the proof of Theorem 5.2.8. The proof of Theorem 5.2.8 can be deduced as follows. In the literature, a prime p satisfying the statement of the Theorem 5.2.8 is often referred to as a prime which satisfies '*Property C*' (see [4] and [12]). We say a prime p satisfies *Property B* if every zero-sum sequence over a group $G \cong \mathbb{Z}_p^2$ of length $2p-1$ containing no proper zero-sum subsequence contains an element with multiplicity $p-1$. In [12] it is shown that if a prime p satisfies Property B then it satisfies Property C. In [25], Reiher proved that every prime number satisfies Property B, which consequently proves Theorem 5.2.8.

The following result does not concern sequences over groups - it is merely a result about integers that we use in Section 5.4.

Lemma 5.2.10. *Let p and x be integers such that $4 \leq x \leq p-1$. Then there exist integers y and z such that $1 \leq y \leq z \leq p-1$ and $xy = z + p$.*

Proof. Let $z = x\lceil p/(x-1)\rceil - p$ and $y = \lceil p/(x-1)\rceil$. Then

$$xy = x\lceil p/(x-1)\rceil = z + p.$$

Now $x - 1 < p$ so $p/(x-1) > 0$ hence $y = \lceil p/(x-1)\rceil \geq 1$. Moreover,

$$z - y = (x-1)\lceil p/(x-1)\rceil - p \geq (x-1)(p/(x-1)) - p = 0$$

which implies $z \geq y$. So it remains to show that $z \leq p-1$. Suppose $p = 5$. Then $4 \leq x \leq p-1$ implies $x = 4$ and hence

$$z = x\lceil p/(x-1)\rceil - p = 4 \times \lceil 5/3\rceil - 5 = 3 < 4 = p - 1.$$

Similarly, supposing $p = 6$, we observe that $z < p - 1$. Suppose $x = p - 1$.

Then, noting that $1 < p/(p-2) < 2$, we deduce

$$z = (p-1)\lceil p/(p-2)\rceil - p = p - 2 < p - 1.$$

Now suppose $p > 6$ and $x < p - 1$. We have that

$$z = x\lceil p/(x-1)\rceil - p < x(p/(x-1) + 1) - p$$

since $\lceil p/(x-1)\rceil < p/(x-1) + 1$. Now

$$\begin{aligned} x(p/(x-1) + 1) - p &= (x - 1 + 1)(p/(x-1)) + x - p \\ &= p + (p/(x-1)) + x - p \\ &= (p/(x-1)) + (x-1) + 1. \end{aligned}$$

Now consider the function

$$\begin{aligned} \psi : [3, p-3] &\longrightarrow \mathbb{R}_{>0} \\ a &\mapsto p/a + a. \end{aligned}$$

We have that

$$z < \psi((x-1)) + 1 \leq \max_{a \in [3,p-3]} \psi(a) + 1.$$

Now we claim that $\max_{a\in[3,p-3]} \psi(a) = \max\{\psi(3), \psi(p-3)\}$. Since ψ is a continuous, real-valued function on the closed interval $[3, p-3]$, by the Extreme Value Theorem we have that ψ achieves its maximum and minimum. In order to show that ψ achieves its maximum at $\max\{\psi(3), \psi(p-3)\}$ we show that ψ has at most one stationary point which is a minimum and therefore it must achieve it maximum on one of its endpoints 3 and $p - 3$. Solving the equation

$$\psi'(a) = 0$$

for $a \in [3, p-3]$ where ψ' denotes the first derivative of ψ gives us the stationary points of ψ. We find that ψ has one stationary point at $a = \sqrt{p}$. We now find the second derivative ψ'' of ψ and note that $\psi''(\sqrt{p}) > 0$. Hence $\sqrt{p}$ is a minimum of ψ. So we come to the conclusion that

$$z < \max\{\psi(3), \psi(p-3)\} + 1 = \max\{p/3 + 4, p/(p-3) + p - 2\}.$$

Now note that, since z is an integer, we have that

$$z \leq \max\{p/3 + 4, \lfloor p/(p-3) + p - 2 \rfloor\}.$$

We have that $p > 6$ implies $p/3 + 4 < p$. It remains to note that $p > 6$ also implies $1 < p/(p-3) < 2$, and hence

$$\lfloor p/(p-3) + p - 2 \rfloor = p - 1.$$

This completes the proof. $\square$

5.3 A result on sequences of length $1 + d^*(G)$

In this section we present a result which provides sufficient conditions for sequences over $\mathbb{Z}_p^3 \oplus \mathbb{Z}_2$, where p is an arbitrary prime number, to contain a non-empty zero-sum subsequence. More precisely, we prove the following:

Proposition 5.3.1. *Fix a prime number p and define $G := \mathbb{Z}_p^3 \oplus \mathbb{Z}_2$. Let $x_1 \cdots x_{4p-2}$ be a sequence over $\mathbb{Z}_p^3$ and $y_1 \cdots y_{4p-2}$ be a sequence over $\mathbb{Z}_2$ such that $y_1 = \cdots = y_r = 1$ and $y_{r+1} = \cdots = y_{4p-2} = 0$ for some integer $r \in \{0, \ldots, 2p\} \cup \{4p-4, 4p-3, 4p-2\}$. Then the sequence*

$$S = (x_1, y_1) \cdots (x_{4p-2}, y_{4p-2})$$

over G is not zero-sum free.

The proof of Proposition 5.3.1 is based on the idea for the proof of Lemma 1.1 in [5].

Proof of Proposition 5.3.1. We split the proof of this proposition into the following four cases: $r \in \{0, \ldots, p\}$, $r \in \{p+1, \ldots, 2p\}$, $r = 4p-4$, and $r \geq 4p-3$.

Case (i): Suppose $r \in \{0, \ldots, p\}$. Define the sequence $T := x_{r+1} \cdots x_{4p-2}$ over $\mathbb{Z}_p^3$. In this case T is a sequence over $\mathbb{Z}_p^3$ of length $4p-2-r \geq 3p-2$. Now note that $D(\mathbb{Z}_p^3) = 3p-2$. Hence T contains a non-empty zero-sum subsequence $U := x_{i_1} \cdots x_{i_t}$. Noting that $y_{i_1} = \cdots = y_{i_t} = 0$, we deduce that U^S is a non-empty zero-sum subsequence of S which completes the proof in this case.

Case (ii): Suppose $r \in \{p+1, \ldots, 2p\}$. Since $r \neq 1$ we can pair the elements $x_1, \ldots, x_{4p-2}$ in the following way. Define $t := \lfloor r/2 \rfloor$ and form the sequence

$$T := (x_1 + x_2) \cdots (x_{2t-1} + y_{2t}) x_{r+1} \cdots x_{4p-2}$$

over $\mathbb{Z}_p^3$. Note that the length of T is equal to $4p - 2 - r + t$. Since $r \leq 2p$, we have that $-r + t \geq -p$. Hence the length of T is greater than or equal to $3p-2$. Since $D(\mathbb{Z}_p^3) = 3p-2$, we deduce that T contains a non-empty zero-sum subsequence $T_1 \cup T_2$ where T_1 and T_2 are subsequences of $(x_1 + x_2) \cdots (x_{2t-1} + y_{2t})$ and $x_{r+1} \cdots x_{4p-2}$ respectively. Suppose $T_1 = x_{i_1} + x_{i_1+1} \cdots x_{i_j} + x_{i_j+1}$ and define the sequence $U := x_{i_1} x_{i_1+1} \cdots x_{i_j} x_{i_j+1}$ over $\mathbb{Z}_p^3$. Since $y_1 + y_2 = \cdots = y_{2t-1} + y_{2t} = y_{r+1} = \cdots = y_{4p-2} = 0$, we have that $U^S \cup T_2^S$ is a zero-sum subsequence of S. This completes the proof in this case.

Case (iii): Suppose $r \geq 4p-3$. Define the embedding $\phi : \mathbb{Z}_p^3 \longrightarrow \mathbb{Z}_p^4$ by

$$(x, y, z) \mapsto (x, y, z, 0).$$

Let $g = (1, 1, 1, 1) \in \mathbb{Z}_p^4$ and consider the sequence

$$U := g + \phi(x_1) \cdots g + \phi(x_{4p-3})$$

over $\mathbb{Z}_p^4$. Since $D(\mathbb{Z}_p^4) = 4p - 3$, we deduce that U contains a non-empty zero-sum subsequence, say

$$V := g + \phi(x_{i_1}) \cdots g + \phi(x_{i_j}).$$

We have that

$$(0, 0, 0, 0) = |V| = jg + \phi(x_{i_1} + \cdots + x_{i_j}) = (j, j, j, j) + \phi(x_{i_1} + \cdots + x_{i_j}).$$

Now since the last component of $\phi(x_{i_1} + \cdots + x_{i_j})$ is 0 we deduce that $j \equiv 0 \pmod p$. This implies that

$$\phi(x_{i_1} + \cdots + x_{i_j}) = (0, 0, 0, 0)$$

from which it follows that

$$T := x_{i_1} \cdots x_{i_j}$$

is a non-empty zero-sum subsequence of $x_1 \cdots x_{4p-3}$ of length j. Now $j \equiv 0 \pmod p$ and $1 \leq j \leq 4p - 3$ implies $j \in \{p, 2p, 3p\}$. We deal with the different values j can take as follows.

- Suppose $j = p$. Then, noting that $4p - 2 = p + D(\mathbb{Z}_p^3)$, we find that Lemma 5.2.5 (i) implies that $x_1 \cdots x_{4p-2}$ contains two non-empty disjoint zero-sum subsequences. Now, Lemma 5.2.4 (i) implies that S is not zero-sum free.

- Suppose $j = 2p$. Then the length of T is even. So Lemma 5.2.4 (ii) implies the subsequence of S obtained by removing (x_{4p-2}, y_{4p-2}) from S is not zero-sum free. Hence S is not zero-sum free.

- Suppose $j = 3p$. Then the length of T is strictly greater than $D(\mathbb{Z}_p^3)$. So Lemma 5.2.5 (ii) implies $x_1 \cdots x_{4p-2}$ contains two non-empty disjoint zero-sum subsequences. Now, Lemma 5.2.4 (i) implies that S is not zero-sum free.

Case (iv): Suppose $r = 4p - 4$. Define ϕ and $g \in \mathbb{Z}_p^4$ as in case (iii) and consider the sequence

$$U := g + \phi(x_1) \cdots g + \phi(x_{4p-4})$$

over $\mathbb{Z}_p^4$. Suppose U is not zero-sum free. Then by following the same argument as in case (iii) we deduce that the sequence $x_1 \cdots x_{4p-4}$ contains a zero-sum subsequence of length j for some $j \in \{p, 2p, 3p\}$, and hence either $x_1 \cdots x_{4p-2}$ contains two non-empty disjoint zero-sum subsequences or $x_1 \cdots x_{4p-4}$ contains a non-empty zero-sum subsequence of even length. Using Lemma 5.2.4 we deduce that S is not zero-sum free in both of the latter cases.

Now suppose that U is a zero-sum free sequence over $\mathbb{Z}_p^4$. Then Proposition 5.2.2 implies that

$$[U] = \mathbb{Z}_p^4 \setminus \{0\}.$$

Let $|x_1 \cdots x_{4p-2}| = (t_1, t_2, t_3)$ for some $t_1, t_2, t_3 \in \mathbb{Z}_p$ and define

$$t := (t_1 + p - 2, t_2 + p - 2, t_3 + p - 2, p - 2) \in \mathbb{Z}_p^4.$$

Now $t \in \mathbb{Z}_p^4 \setminus \{0\}$ so $t \in [U]$. This means there exists a non-empty subsequence

$$V := g + \phi(x_{m_1}) \cdots g + \phi(x_{m_k})$$

of U with $|V| = t$. More precisely,

$$t = |V| = kg + \phi(x_{m_1} + \cdots + x_{m_k}) = (k, k, k, k) + \phi(x_{m_1} + \cdots + x_{m_k}).$$

Now since the last component of $\phi(x_{m_1} + \cdots + x_{m_k})$ is 0, we deduce that

$k \equiv p - 2 \pmod p$ and hence

$$(t_1, t_2, t_3, 0) = \phi(x_{m_1} + \cdots + x_{m_k}).$$

Since $(t_1, t_2, t_3, 0) = \phi(t_1, t_2, t_3)$, we deduce that the subsequence

$$W := x_{m_1} + \cdots + x_{m_k}$$

of $x_1 \cdots x_{4p-4}$ has value is equal to $|x_1 \cdots x_{4p-2}|$. Therefore we can remove W from $x_1 \cdots x_{4p-2}$ to obtain a non-empty zero-sum subsequence T of $x_1 \cdots x_{4p-2}$ of length $4p - 2 - k$. Now $k \equiv p - 2 \pmod p$ and $1 \le k \le 4p - 4$ implies $k \in \{p-2, 2p-2, 3p-2\}$. We deal with these values of k individually.

- Suppose $k = p - 2$. Then the length of T is $3p > D(\mathbb{Z}_p^3)$. Hence Lemma 5.2.5 (ii) implies that $x_1 \cdots x_{4p-2}$ contains two non-empty disjoint zero-sum subsequences. Therefore using Lemma 5.2.4 (i) we deduce that S is not zero-sum free.

- Suppose $k = 2p - 2$. Since k is even in this case, we observe that the sequence T is the union of a subsequence of $x_1 \cdots x_{4p-4}$ of even length and the sequence $x_{4p-3}x_{4p-2}$. Since $y_1 = \cdots = y_{4p-4} = 1$, this implies that T^S is a non-empty zero-sum subsequence of S.

- Suppose $k = 3p - 2$. Then the length of T is p. Hence, noting that $4p-2 = p+D(\mathbb{Z}_p^3)$, we find that Lemma 5.2.5 (i) implies that $x_1 \cdots x_{4p-2}$ contains two non-empty disjoint zero-sum subsequences. Therefore using Lemma 5.2.4 (i) we deduce that S is not zero-sum free.

The proof of the proposition is complete. □

We now make a conjecture which, if proved for a prime number $p \ge 5$, will determine (by Theorem 5.3.3 below) the Davenport constant of the group $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{2p}$. We prove this conjecture for $p = 5$ in Chapter 6, and hence determine the Davenport constant of the group $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$ (see Theorem 6.1).

Conjecture 5.3.2. *Fix a prime number $p \geq 5$ and define $G := \mathbb{Z}_p^3 \oplus \mathbb{Z}_2$. Let $x_1 \cdots x_{4p-2}$ be a sequence over $\mathbb{Z}_p^3$ and $y_1 \cdots y_{4p-2}$ be a sequence over $\mathbb{Z}_2$ such that $y_1 = \cdots = y_r = 1$ and $y_{r+1} = \cdots = y_{4p-2} = 0$ for some even integer $r \in \{2p+2, \ldots, 4p-6\}$. Then the sequence*

$$S = (x_1, y_1) \cdots (x_{4p-2}, y_{4p-2})$$

over G is not zero-sum free.

Theorem 5.3.3. *Let p be a prime number for which Conjecture 5.3.2 holds. Then the equality $D(G) = 1 + d^*(G)$ holds for $G := \mathbb{Z}_p^3 \oplus \mathbb{Z}_2$.*

Proof. We may assume that $p \geq 5$ as the equality is known for all other values of p. Observe that $1+d^*(G) = 4p-2$. We show that any arbitrary sequence S over G of length $4p-2$ contains a non-empty zero-sum subsequence. Suppose

$$S = (x_1, y_1) \cdots (x_{4p-2}, y_{4p-2})$$

for sequences $x_1 \cdots x_{4p-2}$ and $y_1 \cdots y_{4p-2}$ over $\mathbb{Z}_p^3$ and $\mathbb{Z}_2$ respectively. Without loss of generality assume $y_1 = \cdots = y_r = 1$ and $y_{r+1} = \cdots = y_{4p-2} = 0$ for some $r \geq 0$. If $r \in \{0, \ldots, 2p\} \cup \{4p-4, 4p-3, 4p-2\}$ then Proposition 5.3.1 implies S is not zero-sum free. If r is even and $r \in \{2p+2, \ldots, 4p-6\}$ then, since p is a prime number for which Conjecture 5.3.2 holds, we deduce that S is not zero-sum free. It remains to show S is not zero-sum free in the case when r is odd and $r \in \{2p+1, \ldots, 4p-5\}$ in order to complete the proof. Suppose r is an odd integer in the set $\{2p+1, \ldots, 4p-5\}$. Define $t := -|x_1 \cdots x_{4p-2}|$ and consider the sequence

$$T := (x_1, 1) \cdots (x_r, 1), (t, 1), (x_{r+1}, 0) \cdots (x_{4p-3}, 0)$$

over $\mathbb{Z}_p^3 \oplus \mathbb{Z}_2$. We have that T is a sequence of length $4p-2$ and the last components of precisely $r+1$ elements of T are non-zero. If $r \neq 4p-5$ then $r+1$ is an even integer in the set $\{2p+2, \ldots, 4p-6\}$ and so since Conjecture 5.3.2 holds for p, we deduce that T is not zero-sum free. If $r = 4p-5$ then $r+1 = 4p-4$ and so Proposition 5.3.1 implies T is not zero-sum free. Now observe that T is a proper subsequence of the sequence $S \cup (t, 1)$. Hence $S \cup (t, 1)$ contains a proper zero-sum subsequence. Therefore Lemma 5.2.7 implies that S is not zero-sum free. □

5.4 A property about zero-sum free sequences

Given a prime number p, in this section we view $\mathbb{Z}_p^3$ as a 3-dimensional vector space over $\mathbb{Z}_p$ where appropriate. Given a prime number $p > 2$ and a zero-sum free sequence

$$S = (x_1, y_1) \cdots (x_n, y_n)$$

over $\mathbb{Z}_p^3 \oplus \mathbb{Z}_2$, what can we say about the sequence $x_1 \cdots x_n$ over $\mathbb{Z}_p^3$? We find that if n is *sufficiently large* then we can say that $x_1 \cdots x_n$ contains a basis for $\mathbb{Z}_p^3$ over $\mathbb{Z}_p$. Using some fairly elementary techniques we deduce that if $n \geq 3p-1$ then the sequence $x_1 \cdots x_n$ contains a basis. With some more analysis we find that, if we assume $n \geq 4p-2$ and $y_1 = \cdots = y_{3p-3}$, then $x_1 \cdots x_{3p-3}$ contains a basis for $\mathbb{Z}_p^3$ over $\mathbb{Z}_p$. More precisely we show the following two results:

Proposition 5.4.1. *Given a prime number $p > 2$, define $G := \mathbb{Z}_p^3 \oplus \mathbb{Z}_2$. Let*

$$S := (x_1, y_1) \cdots (x_n, y_n)$$

be a zero-sum free sequence over G of length $n = 3p-1$. Then the sequence $x_1 \cdots x_n$ contains a basis for $\mathbb{Z}_p^3$ over $\mathbb{Z}_p$.

Proposition 5.4.2. *Given a prime number $p > 2$, define $G := \mathbb{Z}_p^3 \oplus \mathbb{Z}_2$. Let*

$$S := (x_1, y) \cdots (x_{3p-3}, y)(x_{3p-2}, y_{3p-2}) \cdots (x_n, y_n)$$

be a zero-sum free sequence over G of length $n = 4p - 2$. (Note the second components of the first $3p-3$ terms are equal.) Then the sequence $x_1 \cdots x_{3p-3}$ contains a basis for $\mathbb{Z}_p^3$ over $\mathbb{Z}_p$.

We shall prove Proposition 5.4.1 first, for which we need the following preliminaries.

Theorem 5.4.3 (Theorem 3.10 (2) in [20])**.** *If S is a spanning set for a finite dimensional vector space V over a field $\mathbb{F}$ then S contains a basis for V over $\mathbb{F}$.*

Lemma 5.4.4. *Let S be a sequence over $\mathbb{Z}_p^3$ such that*

$$\langle S \rangle = \mathbb{Z}_p^3 .$$

Then S contains a basis for $\mathbb{Z}_p^3$ over $\mathbb{Z}_p$.

Proof. Let $S = x_1 \cdots x_n$ where $x_i \in \mathbb{Z}_p^3$ for all $i \in [1, n]$. Then $\langle S \rangle = \mathbb{Z}_p^3$ implies $\mathrm{Span}_{\mathbb{Z}_p}\{x_1, \dots, x_n\} = \mathbb{Z}_p^3$. Therefore by Theorem 5.4.3 we deduce that $\{x_1, \dots, x_n\}$ contains a basis for $\mathbb{Z}_p^3$ over $\mathbb{Z}_p$. □

Proof of Proposition 5.4.1. We claim that

$$\langle S \rangle = G.$$

In order to prove the claim suppose for a contradiction that $\langle S \rangle = H$ for some non-trivial proper subgroup H of G (note that $\langle S \rangle \neq \{0\}$ else S contains the element $(0, 0) \in \mathbb{Z}_p^3 \oplus \mathbb{Z}_2$). Using Theorem 3.2, we find that H is isomorphic to one of $\mathbb{Z}_2$, $\mathbb{Z}_p$, $\mathbb{Z}_{2p}$, $\mathbb{Z}_p^2$, $\mathbb{Z}_p \oplus \mathbb{Z}_{2p}$, or $\mathbb{Z}_p^3$. Hence by Theorem 2.1.1 and

Theorem 2.1.2, we have that $D(H) = 1 + d^*(H)$. Moreover, we deduce that

$$d^*(H) \in \{1, p-1, 2p-1, 2p-2, 3p-2, 3p-3\}$$

and hence

$$D(H) < 3p - 1.$$

Hence S is a sequence over H of length at least $D(H)$ which contradicts the assumption that S is zero-sum free. This proves the claim. Observe that the claim implies

$$\langle x_1 \cdots x_n \rangle = \mathbb{Z}_p^3.$$

It remains to apply Lemma 5.4.4 to complete the proof. □

In order to prove Proposition 5.4.2 we require the following result.

Theorem 5.4.5. *Given a prime number $p > 2$, define $G := \mathbb{Z}_p^3 \oplus \mathbb{Z}_2$. Let*

$$S := (x_1, y) \cdots (x_{3p-3}, y)(x_{3p-2}, y_{3p-2}) \cdots (x_n, y_n)$$

be a zero-sum free sequence over G of length $n = 4p - 2$. Then

$$\langle S' \rangle = \mathbb{Z}_p^3$$

where $S' := x_1 \cdots x_{3p-3}$.

Proof. We claim that the sequence $x_1 \cdots x_{4p-2}$ over $\mathbb{Z}_p^3$ does not contain a non-empty zero-sum subsequence of length at most p. Suppose for a contradiction that $x_1 \cdots x_{4p-2}$ contains a non-empty zero-sum subsequence of length at most p. Then since $4p - 2 = p + D(\mathbb{Z}_p^3)$, we find that Lemma 5.2.5 (i) tells us that $x_1 \cdots x_{4p-2}$ contains two non-empty disjoint zero-sum subsequences. Hence Lemma 5.2.4 (i) implies that S is not zero-sum free which contradicts our assumption.

We know that $\langle S' \rangle$ is a subgroup of $\mathbb{Z}_p^3$, hence $\langle S' \rangle \cong \mathbb{Z}_p^n$ for some $n \in \{1, 2, 3\}$. We show that $n \notin \{1, 2\}$.

Suppose for a contradiction that $\langle S' \rangle \cong \mathbb{Z}_p$. Then, since $3p - 3 > p$ and $p = D(\mathbb{Z}_p)$, we have that S' contains a non-empty zero-sum subsequence of length at most p. This implies that $x_1 \cdots x_{4p-2}$ contains a zero-sum subsequence of length at most p which produces a contradiction.

Suppose for a contradiction that $\langle S' \rangle \cong \mathbb{Z}_p^2$. Using Theorem 5.2.8 we deduce that S' must be of the form

$$S' = a \ldots ab \ldots bc \ldots c$$

for some pairwise distinct elements $a, b, c \in \mathbb{Z}_p^3$ each having multiplicity $p - 1$ in S'. Hence

$$\mathbb{Z}_p^2 \cong \mathrm{Span}_{\mathbb{Z}_p}\{a, b, c\}.$$

By Theorem 5.4.3, we deduce that the set $\{a, b, c\}$ contains a basis for $\mathrm{Span}_{\mathbb{Z}_p}\{a, b, c\}$ over $\mathbb{Z}_p$. Without loss of generality suppose $\{a, b\}$ is a basis for $\mathrm{Span}_{\mathbb{Z}_p}\{a, b, c\}$ over $\mathbb{Z}_p$. Then there exist $q_1, q_2 \in \{0, \ldots, p-1\}$ such that

$$q_1 a + q_2 b + c = 0. \tag{5.1}$$

Consider the sequence

$$Q := a \cdots ab \cdots bc$$

in which the multiplicity of a is q_1, the multiplicity of b is q_2 and multiplicity of c is 1. Note that equation (5.1) implies that Q is a non-empty zero-sum subsequence of S' of length $q_1 + q_2 + 1$. Define $l := q_1 + q_2 + 1$. We shall complete the rest of the proof by examining the range of values l can take.

Case (i): Suppose $l \in \{1, \ldots, p\}$. Then Q is a non-empty zero-sum subsequence of S' of length at most p. This is a contradiction.

Case (ii): Suppose l is even. Then Lemma 5.2.4 (ii) implies S is not zero-sum free which is a contradiction.

Case (iii): Suppose l is an odd integer strictly greater than p. Since

$q_1, q_2 \in \{0, \ldots, p-1\}$, we have that

$$p + 1 \leq l \leq 2p - 1$$

which implies

$$1 \leq l - p \leq p - 1.$$

Now since both l and p are odd we must have that $l - p = x$ for some even integer $1 \leq x \leq p - 1$. Now we proceed by examining the cases $x = 2$ and $x > 2$.

Suppose $x = 2$. Then

$$l \equiv 2 \pmod{p}.$$

Suppose $(p-1)q_1 \equiv \alpha \pmod{p}$ and $(p-1)q_2 \equiv \beta \pmod{p}$ for some integers $\alpha, \beta \in \{0, \ldots, p-1\}$ and consider the sequence

$$T := a \cdots ab \cdots bc \cdots c$$

in which the multiplicity of a is α, the multiplicity of b is β and the multiplicity of c is $p - 1$. By equation (5.1) we have

$$|T| = \alpha a + \beta b + (p-1)c = (p-1)(q_1 a + q_2 b + c) = 0.$$

Moreover, the length of T is equal to

$$\alpha + \beta + p - 1 \equiv (p-1)l \equiv 2p - 2 \pmod{p}.$$

Now $p - 1 \leq \alpha + \beta + p - 1 \leq 3p - 3$ implies $\alpha + \beta + p - 1 = 2p - 2$. Hence T is a non-empty zero-sum subsequence of S' of even length. Therefore Lemma 5.2.4 (ii) implies S is not zero-sum free which contradicts our assumption.

Suppose $x > 2$. Then by Lemma 5.2.10 there exist integers y and z such that $1 \leq y \leq z \leq p - 1$ and $xy = z + p$. Suppose $yq_1 \equiv \alpha' \pmod{p}$ and

$yq_2 \equiv \beta' \pmod{p}$ for some $\alpha', \beta' \in \{0, \ldots, p-1\}$ and consider the sequence

$$T' := a \cdots ab \cdots bc \cdots c$$

in which the multiplicity of a is α', the multiplicity of b is β' and the multiplicity of c is y. By equation (5.1) we have

$$|T'| = \alpha' a + \beta' b + yc = y(q_1 a + q_2 b + c) = 0.$$

Moreover, since $l \equiv x \pmod{p}$, the length of T' is equal to

$$\alpha' + \beta' + y \equiv yl \equiv xy \equiv z \pmod{p}.$$

Now $1 \leq \alpha' + \beta' + y \leq 2p - 2 + z$ implies the length of T' is either z or $z + p$. Suppose the length of T' is z. Then, since $z < p$, we deduce that T' is a non-empty zero-sum subsequence of S' of length at most p. This produces a contradiction. Suppose the length of T' is $z + p$. Then, since $z + p = xy$ and x is even, we deduce that T' is a non-empty zero-sum subsequence of S' of even length. Hence Lemma 5.2.4 (ii) implies S is not zero-sum free which contradicts our assumption. This completes the proof. $\square$

Proof of Proposition 5.4.2. By Theorem 5.4.5 we have $\langle x_1 \cdots x_{3p-3} \rangle = \mathbb{Z}_p^3$. Hence the result follows from Lemma 5.4.4. $\square$

Chapter 6

The Davenport constant of $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$

The aim of this chapter is to determine the Davenport constant of the group $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$. As discussed in Chapter 2 it is conjectured that the equality

$$D(G) = 1 + d^*(G) \tag{6.1}$$

holds for all groups G of rank 3. The reason we are interested in finding the Davenport constant of the group $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$ is because it is the smallest group of rank 3 for which (6.1) is not known to hold (see Theorem 2.2.1). We show that

$$D(\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}) = 1 + d^*(\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}) = 18.$$

A natural attempt to prove the equality (6.1) for $G := \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$ would be to use Lemma 4.2.3 as follows. Define

$$H := \{(0, 0, z) \in G \mid z \equiv 0 \bmod 5\}$$

and $F := G/H$ and note that $H \cong \mathbb{Z}_2$. Then using Lemma 4.2.3 it is sufficient to show that every sequence of length 18 over G contains 2 disjoint F-zero-sum subsequences with respect to the canonical homomorphism from G to F. Since $F \cong \mathbb{Z}_5^3$, this is equivalent to showing that every sequence over $\mathbb{Z}_5^3$ of length 18 contains two non-empty disjoint zero-sum subsequences. Unfortunately, the sequence of length 18 over $\mathbb{Z}_5^3$ which contains each of the elements $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ with multiplicity 4, and each of the elements $(0,1,1)$, $(1,1,0)$ and $(1,0,1)$ with multiplicity 2 does not contain two non-empty disjoint zero-sum subsequences; it is relatively simple to verify this by writing a program in the computer algebra system Magma (see Corollary 7.2.12). This means we cannot use Lemma 4.2.3 as described above in order show $D(G) = 18$. We use a different approach to show $D(G) = 18$ as described below.

Theorem 6.1. *The equality $D(G) = 1 + d^*(G)$ holds for $G := \mathbb{Z}_5^3 \oplus \mathbb{Z}_2$.*

Proof. By Theorem 5.3.3 it is sufficient to prove Conjecture 5.3.2 for the prime number 5 in order to prove the result. Let $x_1 \cdots x_{18}$ be a sequence over $\mathbb{Z}_5^3$ and $y_1 \cdots y_{18}$ be a sequence over $\mathbb{Z}_2$ such that $y_1 = \cdots y_r = 1$ and $y_{r+1} = \cdots = y_{18} = 0$ for some $r \in \{12, 14\}$. Then we claim that the sequence

$$S = (x_1, y_1) \cdots (x_{18}, y_{18})$$

over G is not zero-sum free. Suppose for a contradiction that S is zero-sum free. We start by making a few observations using this assumption.

Firstly observe that there does not exist a subsequence Z of $x_{r+1} \cdots x_{18}$ such that $-|Z| = |T|$ for some non-empty subsequence T of $x_1 \cdots x_r$ of even length. Indeed, if this is the case then $(T \cup Z)^S$ is a non-empty zero-sum subsequence of S which contradicts the assumption that S is zero-sum free. Note that this observation implies that the sequence $x_1 \cdots x_r$ does not contain a non-empty zero-sum subsequence of even length.

Secondly, observe that the sequence $x_1 \cdots x_{18}$ does not contain a non-empty zero-sum subsequence of length at most 5 or length at least 14. Indeed, if this is the case then, noting that $18 = 5 + D(\mathbb{Z}_5^3)$, we deduce using Lemma 5.2.5 that $x_1 \cdots x_{18}$ contains two non-empty disjoint zero-sum subsequences, hence S is not zero-sum free by Lemma 5.2.4 (i) which is a contradiction.

Now we determine three elements of the sequence $x_1 \cdots x_r$ up to isomorphism. Since $y_1 = \cdots = y_{12}$, we deduce that, viewing $\mathbb{Z}_5^3$ as a vector space over $\mathbb{Z}_5$, Proposition 5.4.2 implies the sequence $x_1 \cdots x_r$ contains a basis for $\mathbb{Z}_5^3$ over $\mathbb{Z}_5$. Without loss of generality assume this basis comprises the elements x_1, x_2 and x_3. Now fix a representation for $\mathbb{Z}_3^3$ which comprises of representing all elements of $\mathbb{Z}_3^3$ as the set of coordinate vectors with respect to the ordered bases $\{x_1, x_2, x_3\}$. This means that $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$ and $x_3 = (0, 0, 1)$.

Now we claim that the sequence $x_1 \cdots x_6$ is zero-sum free. Indeed, this is the case else $x_1 \cdots x_r$ contains a non-empty zero-sum subsequence of even length or length at most 5 both of which produce a contradiction. The next step of the proof involves using a computer program to determine all possibilities for the sequence $x_1 \cdots x_6$. We shall refer to this computer program as *CPF6*. The program CPF6 generates all zero-sum free sequences over $\mathbb{Z}_5^3$ of length 6 containing the subsequence $(1, 0, 0)(0, 1, 0)(0, 0, 1)$. We find that there are 208334 such sequences. We describe CPF6 in detail in Section 8.2.3.

The idea now is to compute a set of possibilities for the sequence $x_1 \cdots x_{18}$ by *extending* each possibility for $x_1 \cdots x_6$ generated by CPF6. We do this as follows. Suppose $r = 14$. In this case we create a computer program called *CPF14* which takes as input a possibility U for $x_1 \cdots x_6$ generated by CPF6 and outputs all sequences X over $\mathbb{Z}_5^3$ of length 14 subject to the following conditions:

- The sequence X contains U as a subsequence;
- The sequence X does not contain a non-empty zero-sum subsequence of even length or length at most 5.

Inputting each sequence generated by CPF6 in the program CPF14, we obtain a set of possibilities for the sequence $x_1 \cdots x_{14}$. We describe CPF14 in Section 8.2.4. We then create a computer program which we refer to as *CPF14EXT* which takes as input a possibility X for $x_1 \cdots x_{14}$ generated by CPF14 and outputs all sequences $X \cup z_{15} \cdots z_{18}$ over $\mathbb{Z}_5^3$ of length 18 subject to the following conditions:

- The sequence $X \cup z_{15} \cdots z_{18}$ does not contain a non-empty zero-sum subsequence of length at most 5 or length at least 14;
- The inverse of the value of each of the following sequences does not occur as the value of some subsequence of X of even length: z_{15}, z_{16}, z_{17}, z_{18}, $z_{15}z_{16}$, $z_{15}z_{17}$, $z_{15}z_{18}$, $z_{16}z_{18}$, $z_{17}z_{18}$, $z_{15}z_{16}z_{17}$, $z_{16}z_{17}z_{18}$, $z_{15}z_{16}z_{17}z_{18}$.

Inputting each sequence generated by CPF14 in the program CPF14EXT, we obtain a set of possibilities for the sequence $x_1 \cdots x_{18}$. We describe CPF14EXT in Section 8.2.5. We find that CPF14EXT completes its run without producing a possibility for $x_1 \cdots x_{18}$. This means that the assumption that S is zero-sum free cannot hold.

Now suppose $r = 12$. To produce a contradiction in this case we create computer programs *CPF12* and *CPF12EXT* which work analogously to CPF14 and CPF14EXT respectively - see Section 8.2.6 and Section 8.2.7. We find that the total running time for CPF12 with the first 100 outputs of CPF6 is approximately 4000 minutes. This is considerably longer as compared to the total running time for CPF14 with the first 100 outputs of CPF6 which is approximately 9 minutes. For this reason, we input the sequence

$$U := (1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,0,1)$$

output from CPF6 into CPF12 and obtain as output all sequences over $\mathbb{Z}_3^3$ of length 12 containing the subsequence U and no non-empty zero-sum subsequence of even length or length at most 5. We then run CPF12EXT on all of these outputs and find that CPF12EXT completes its runs without producing an output. From this we deduce that the sequence $x_1 \cdots x_{12}$ cannot contain an element with multiplicity 4. Indeed, suppose $x_1 \cdots x_{12}$ contains an element with multiplicity 4, say $x_9 = x_{10} = x_{11} = x_{12}$. Then, since $D(\mathbb{Z}_5) = 5$ implies

$$\langle x_8 \cdots x_{12} \rangle \ncong \mathbb{Z}_5,$$

we deduce that the elements x_8 and x_9 are linearly independent if we view $\mathbb{Z}_5^3$ as a vector space over $\mathbb{Z}_5$. Now Theorem 5.4.5 implies that $\langle x_1 \cdots x_{12} \rangle \cong \mathbb{Z}_5^3$. Hence there exists $i \in \{1, \ldots, 7\}$ such that the set $\{x_9, x_8, x_i\}$ forms a basis for $\mathbb{Z}_5^3$ over $\mathbb{Z}_5$. Representing the elements of $\mathbb{Z}_5^2$ as coordinate vectors with respect to this basis we can assume that $x_1 \cdots x_{12}$ contains U as a subsequence. However we have previously deduced that if this is the case then there exist no possibilities for $x_1 \ldots x_{18}$ which is a contradiction. Now we run CPF12 over all outputs of CPF6 with the additional condition that $x_1 \cdots x_{12}$ does not contain an element with multiplicity 4 to obtain a set P of possibilities for $x_1 \cdots x_{12}$. We then run CPF12EXT over all sequences in P and find that there do not exist any possibilities for the sequence $x_1 \cdots x_{18}$. This implies that S cannot be zero-sum free and hence completes the proof. $\square$

Chapter 7

An upper bound on the Davenport constant of $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d}$

7.1 Motivation

Given that the equality $D(G) = 1 + d^*(G)$ holds for $G = \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$, it is natural to ask whether the equality holds for all groups G of the form $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d}$ where $d \in \mathbb{N}$. In this chapter we show that

$$D(\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d}) \leq 1 + d^*(\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d}) + 4.$$

In order to prove this upper bound, we need some results about sequences over the group $\mathbb{Z}_5^3$. We shall detail these results in Section 7.2. We present the proof of the upper bound in Section 7.3.

7.2 Some results about $\mathbb{Z}_5^3$

In this section, we present a selection of results about sequences over $\mathbb{Z}_5^3$. We shall use some of these results to prove the upper bound on $D(\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d})$ stated in the previous section.

Definition 7.2.1. Let $k \in \mathbb{N}$. For a group G, define $D^k(G)$ to be the smallest $n \in \mathbb{N}$ such that every sequence over G of length n contains a non-empty zero-sum subsequence of length at most k.

Remark 7.2.2. For a finite abelian group G, we shall write $D^k(G) = \infty$ if for every $n \in \mathbb{N}$, we can find a sequence over G of length n which does not contain a non-empty zero-sum subsequence of length at most k.

Lemma 7.2.3. *For all finite abelian groups G, we have $D^k(G) = \infty$ for $1 \leq k < \exp(G)$.*

Proof. Fix $1 \leq k < \exp(G)$ and $n \in \mathbb{N}$. We claim that we can find a sequence over G of length n which does not contain a non-empty zero-sum subsequence of length at most k. Let $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ for some $1 < n_1 \mid \cdots \mid n_r$. Define g to be the r-tuple in G with all entries equal to 1. Define S to be the sequence over G of length n consisting of n copies of g. Then it is easy to see that S does not contain a non-empty zero-sum subsequence of length at most k. $\square$

Theorem 7.2.4 (Theorem 5.8.3 in [14])**.** *Let $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ where $1 \leq n_1 \mid n_2$ be a finite abelian group of rank at most 2. Then*

$$D^k(G) = 2n_1 + n_2 - 2,$$

where $k := \exp(G)$.

Definition 7.2.5. A sequence S over a group G is said to be *square-free* if the multiplicity of g in S is at most 1 for all $g \in G$.

Definition 7.2.6. Let $k \in \mathbb{N}$. For a group G, define $D^{k^*}(G)$ to be the smallest $n \in \mathbb{N}$ such that every square-free sequence over G of length n contains a non-empty zero-sum subsequence of length at most k.

Remark 7.2.7. Let G be a finite abelian group. Then $D^{k^*}(G) \leq D^k(G)$ for all $k \in \mathbb{N}$. Additionally, $D^{k^*}(G) \leq |G|$ for all $k \in \mathbb{N}$. Furthermore, if $k_1 \leq k_2$, then $D^{k_2}(G) \leq D^{k_1}(G)$ and $D^{k_2{}^*}(G) \leq D^{k_1{}^*}(G)$.

Lemma 7.2.8. *For all finite abelian groups G, we have $D^{1^*}(G) = |G|$.*

Proof. It is easy to see that $D^{1^*}(G) \leq |G|$ as $|G|$ distinct elements in G always include 0_G. In order to see $|G| \leq D^{1^*}(G)$, note that the sequence over G of length $|G|-1$ containing all elements of G except for 0_G does not contain a zero-sum subsequence of length 1. □

The following is the main result of this section. In the proof of this result, we will view $\mathbb{Z}_5^3$ as a vector space over $\mathbb{Z}_5$ where needed.

Theorem 7.2.9. *Define $G := \mathbb{Z}_5^3$. Then*

$$\begin{aligned}
D^{1^*}(G) &= 125, & D^k(G) &= \infty \text{ for } 1 \leq k \leq 4,\\
D^{5^*}(G) &= 15, & D^5(G) &= 33,\\
D^{6^*}(G) &= 14, & D^6(G) &= 24,\\
D^{7^*}(G) &= 13, & D^7(G) &= 19,\\
D^{k^*}(G) &= 12 \text{ for } k \geq 8, & D^8(G) &= 18,\\
& & D^9(G) &= 17,\\
& & D^{10}(G) &= 15,\\
& & D^{11}(G) &= 14,\\
& & D^{12}(G) &= 14,\\
& & D^k(G) &= 13, \text{ for } k \geq 13.
\end{aligned}$$

Proof. Note that $D^{1^*}(G) = 125$ and $D^k(G) = \infty$ for $1 \leq k \leq 4$ follow from Lemma 7.2.8 and Lemma 7.2.3, respectively.

The result $D^5(G) = 33$ can be deduced as a special case of Theorem 1.7 in [11].

We claim that $15 \leq D^{5^*}(G)$, $14 \leq D^{6^*}(G)$, and $13 \leq D^{7^*}(G)$. In order to prove this claim, we find square-free sequences over G of lengths 14, 13, and 12, which do not contain a non-empty zero-sum subsequence of length at most 5, at most 6, and at most 7 respectively. They are as follows:

$$\begin{aligned}
\text{Length 14: } & (1,0,0)(0,1,0)(0,0,1)(2,0,0)(1,1,0)(2,1,0)(3,1,0)\\
& (1,0,1)(2,0,1)(3,0,1)(0,1,1)(1,1,1)(2,1,1)(3,1,1)\\
\text{Length 13: } & (1,0,0)(0,1,0)(0,0,1)(2,0,0)(1,1,0)(2,1,0)(3,1,0)\\
& (1,0,1)(2,0,1)(3,0,1)(0,1,1)(1,1,1)(2,1,1)\\
\text{Length 12: } & (1,0,0)(0,1,0)(0,0,1)(2,0,0)(1,1,0)(2,1,0)(3,1,0)\\
& (1,0,1)(2,0,1)(3,0,1)(0,1,1)(1,1,1)
\end{aligned}$$

We build computer programs, which we name *CPF5L**, *CPF6L**, and *CPF7L**, to check that the above sequences have the previously described property. We shall detail these computer programs in Section 8.2.15.

We show that $24 \leq D^6(G)$, $19 \leq D^7(G)$, $18 \leq D^8(G)$, $17 \leq D^9(G)$, $15 \leq D^{10}(G)$, and $14 \leq D^{12}(G) \leq D^{11}(G)$ in a similar way. More precisely, we find sequences over G over lengths 23, 18, 17, 16, 14, and 13, which do not contain a non-empty zero-sum subsequence of length at most 6, at most 7, at most 8, at most 9, at most 10, and at most 12 respectively. They are as follows:

Length 23: $(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$
$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(4,1,1)(4,1,1)(4,1,1)(4,1,1)$
$(1,1,0)(1,1,0)(1,1,0)(1,0,1)(1,0,1)(1,0,1)(0,1,1)$

Length 18: $(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$
$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(4,1,1)(4,1,1)(4,1,1)(4,1,1)$
$(1,1,0)(1,1,0)$

Length 17: $(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$
$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(4,1,1)(4,1,1)(4,1,1)(4,1,1)$
$(1,1,0)$

Length 16: $(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$
$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(4,1,1)(4,1,1)(4,1,1)(4,1,1)$

Length 14: $(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$
$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(1,1,1)(1,1,1)$

Length 13: $(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$
$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(1,1,1)$

We build computer programs, which we name *CPF6L*, *CPF7L*, *CPF8L*, *CPF9L*, *CPF10L*, and *CPF12L* to check that the above sequences have the previously described property. We shall detail these computer programs in Section 8.2.16.

We now claim that $D^{5^*}(G) \leq 15$. Let $S = g_1 \cdots g_{15}$ be an arbitrary square-free sequence over G of length 15. Suppose for a contradiction that S does not contain a non-empty zero-sum subsequence of length at most 5. If $\langle S \rangle \cong \mathbb{Z}_5$ then we obtain a contradiction using $D(\mathbb{Z}_5) = 5$. If $\langle S \rangle \cong \mathbb{Z}_5^2$, then we obtain a contradiction since Theorem 7.2.4 implies $D^5(\mathbb{Z}_5^2) = 13$. Hence $\langle S \rangle = \mathbb{Z}_5^3$. Therefore, Lemma 5.4.4 implies that S contains a basis for $\mathbb{Z}_5^3$ over $\mathbb{Z}_5$. Representing the elements of $\mathbb{Z}_5^3$ with respect to this basis we can assume that S contains $(1,0,0)(0,1,0)(0,0,1)$ as a subsequence. We now build a computer program, which we name *CPF5U**, to generate all square-free sequences over G of length 15 containing $(1,0,0)(0,1,0)(0,0,1)$ as a subsequence and no non-empty zero-sum subsequence of length at most 5. We detail CPF5U* in Section 8.2.8. We find that CPF5U* does not generate any such sequences. This is a contradiction and the claim is proved. We conclude that $D^{5^*}(G) = 15$.

Similarly, we show that $D^{6^*}(G) \leq 14$ and $D^{7^*}(G) \leq 13$. We build computer programs, which we name *CPF6U** and *CPF7U**, to generate all square-free sequences over G of length 14 and 13 respectively, containing $(1,0,0)(0,1,0)(0,0,1)$ as a subsequence and no non-empty zero-sum subsequence of length at most 6 and at most 7 respectively. We detail these computer programs in Section 8.2.8. We find that these programs do not generate any output. Hence we conclude that $D^{6^*}(G) = 14$ and $D^{7^*}(G) = 13$.

Now we show that $D^{8^*}(G) = 12$. Consider the following sequence over G of length 11 :

$$\begin{aligned}&(1,0,0)(0,1,0)(0,0,1)(2,0,0)(1,1,0)(1,0,1)(2,1,0)(3,1,0)\\&(2,0,1)(3,0,1)(4,1,1).\end{aligned} \tag{7.1}$$

We build a computer program, which we name *CPF8L**, which shows that the above sequence is zero-sum free. We detail CPF8L* in Section 8.2.15. This shows that $12 \leq D^{8^*}(G)$. We now claim that $D^{8^*}(G) \leq 12$. Let $S = g_1 \cdots g_{12}$ be an arbitrary square-free sequence over G of length 12. Suppose for a contradiction that S does not contain a non-empty zero-sum subsequence of length at most 8. If $\langle S \rangle \cong \mathbb{Z}_5$ then we obtain a contradiction using $D(\mathbb{Z}_5) = 5$. If $\langle S \rangle \ncong \mathbb{Z}_5$, then S contains at least two linearly independent elements. Therefore, we can assume that S contains $(1,0,0)(0,1,0)$ as a subsequence. We now build a computer program, which we name *CPF8U**, to generate all square-free sequences over G of length 12 containing $(1,0,0)(0,1,0)$ as a subsequence and no non-empty zero-sum subsequence of length at most 8. We detail CPF8U* in Section 8.2.8. We find that CPF8U* does not generate any such sequences. Hence the claim is proved and we conclude that $D^{8^*}(G) = 12$.

Let $k \geq 9$ be an integer. By the sequence (7.1) above, we have $12 \leq D^{k^*}(G)$. Now note that $D^{k^*}(G) \leq D^{8^*}(G) = 12$. Hence $D^{k^*}(G) = 12$.

Now we show that $D^6(G) \leq 24$. Let $S = g_1 \cdots g_{24}$ be an arbitrary sequence over G of length 24. Suppose for a contradiction that S does not contain a non-empty zero-sum subsequence of length at most 6. Since $D^{6^*}(G) = 14$, we have that S is not square-free. Without loss of generality, assume $g_1 = g_2$. Similarly, the sequence $g_3 \cdots g_{24}$ of length 22 cannot be square-free. Without loss of generality assume $g_3 = g_4$. Continuing this process we can assume that S is of the following form:

$$S = g_1 g_1 g_3 g_3 g_5 g_5 g_7 g_7 g_9 g_9 g_{11} g_{11} g_{13} \cdots g_{24}.$$

Define $S' := g_1 g_1 g_3 g_3 g_5 g_5 g_7 g_7 g_9 g_9 g_{11} g_{11} g_{13}$. If $\langle S' \rangle \cong \mathbb{Z}_5$, then we obtain a contradiction using $D(\mathbb{Z}_5) = 5$. If $\langle S' \rangle \cong \mathbb{Z}_5^2$, then we obtain contradiction using $D^5(\mathbb{Z}_5^2) = 13$. Hence $\langle S' \rangle \cong \mathbb{Z}_5^3$. Therefore S' contains a basis B for $\mathbb{Z}_5^3$ over $\mathbb{Z}_5$. Depending on whether B contains the element g_{13}, we deduce that S must have one of the following two forms; either

$$S = (1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)(0,0,1) g_7 g_7 g_9 g_9 g_{11} g_{11} g_{13} \cdots g_{24}$$

or

$$S = (1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1) g_6 g_6 g_8 g_8 g_{10} g_{10} g_{12} g_{12} g_{14} \cdots g_{24}.$$

Now we build two computer programs, which we name *CPF6U1* and *CPF6U2*, to generate all sequences over G of the first and second form respectively, which do not contain a non-empty zero-sum subsequence of length at most 6. We shall describe these programs in Section 8.2.9. We find that both of these programs do not generate an output. This is a contradiction. We conclude that $D^6(G) = 24$.

Similarly, we show that $D^7(G) \leq 19$. Let $S = g_1 \cdots g_{19}$ be an arbitrary sequence over G of length 19. Suppose for a contradiction that S does not contain a non-empty zero-sum subsequence of length at most 7. Using $D^{7^*}(G) = 13$, we can assume that S is of the following form:

$$S = g_1 g_1 g_3 g_3 g_5 g_5 g_7 g_7 g_9 \cdots g_{19}. \tag{7.2}$$

Define $S' := g_1 g_1 g_3 g_3 g_5 g_5 g_7 g_7 g_9 g_{10} g_{11} g_{12} g_{13}$. As in the previous case, we can narrow our focus to the case $\langle S' \rangle \cong \mathbb{Z}_5^3$. In this case, we deduce that S' contains a basis for $\mathbb{Z}_5^3$ over $\mathbb{Z}_5$, and depending on whether this basis contains 0, 1, 2 or 3 elements from the sequence $g_9 g_{10} g_{11} g_{12} g_{13}$, we deduce that S must have one of the following forms; either

$$S = (1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)(0,0,1) g_7 g_7 g_9 \cdots g_{19} \tag{7.3}$$

or

$$S = (1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1) g_6 g_6 g_8 g_8 g_{10} \cdots g_{19} \tag{7.4}$$

or

$$S = (1,0,0)(1,0,0)(0,1,0)(0,0,1) g_5 g_5 g_7 g_7 g_9 g_9 g_{11} \cdots g_{19} \tag{7.5}$$

or

$$S = (1,0,0)(0,1,0)(0,0,1) g_4 g_4 g_6 g_6 g_8 g_8 g_{10} g_{10} g_{12} \cdots g_{19}. \tag{7.6}$$

Now we build four computer programs to generate all sequences over G of each of the four forms above respectively, which do not contain a non-empty zero-sum subsequence of length at most 7. We shall name these programs *CPF7U1*, *CPF7U2*, *CPF7U3*, and *CPF7U4* and detail them in Section 8.2.10. We find that none of these programs generate an output, which produces a contradiction. We conclude that $D^7(G) = 19$.

Very similarly, we show that $D^8(G) \leq 18$. Let $S = g_1 \cdots g_{18}$ be an arbitrary sequence over G of length 18. Suppose for a contradiction that S does not contain a non-empty zero-sum subsequence of length at most 8. Using $D^{8^*}(G) = 12$, we can assume that S is of the form (7.2) with g_{19} removed. Define $S' := g_1g_1g_3g_3g_5g_5g_7g_7g_9g_{10}g_{11}g_{12}g_{13}$ and consider the case $\langle S' \rangle \cong \mathbb{Z}_5^3$. In this case S must be of the form (7.3), or (7.4), or (7.5), or (7.6), with g_{19} removed. Similar to before, we build four computer programs, which we name *CPF8U1*, *CPF8U2* , *CPF8U3*, and *CPF8U4*, to generate all sequences over G of length 18 of the four forms respectively, which do not contain a non-empty zero-sum subsequence of length at most 8. We detail these computer programs in Section 8.2.11. We find that none of the programs produce an output. We conclude that $D^8(G) = 18$.

We show $D^9(G) \leq 17$ in the same way. Let $S = g_1 \cdots g_{17}$ be an arbitrary sequence over G of length 17. Suppose for a contradiction that S does not contain a non-empty zero-sum subsequence of length at most 9. Using $D^{9^*}(G) = 12$, we can assume that S is of the following form:

$$S = g_1g_1g_3g_3g_5g_5g_7 \cdots g_{17}.$$

Define $S' := g_1g_1g_3g_3g_5g_5g_7 \cdots g_{13}$ and consider the case $\langle S' \rangle \cong \mathbb{Z}_5^3$. In this case, we deduce that S must have one of the following forms; either

$$S = (1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)(0,0,1)g_7 \cdots g_{17}$$

or

$$S = (1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)g_6g_6g_8 \cdots g_{17}$$

or

$$S = (1,0,0)(1,0,0)(0,1,0)(0,0,1)g_5g_5g_7g_7g_9 \cdots g_{17}$$

or

$$S = (1,0,0)(0,1,0)(0,0,1)g_4g_4g_6g_6g_8g_8g_{10}\cdots g_{17}.$$

We build four computer programs, which we name *CPF9U1*, *CPF9U2*, *CPF9U3*, and *CPF9U4*, to generate all sequences over G of length 17 of the four forms above respectively, which do not contain a non-empty zero-sum subsequence of length at most 9. We detail these computer programs in Section 8.2.12. We find that none of the programs produce an output. This is a contradiction. Therefore $D^9(G) = 17$.

The proof for $D^{10}(G) \leq 15$ is analogous. Let $S = g_1 \cdots g_{15}$ be an arbitrary sequence over G of length 15. Suppose for a contradiction that S does not contain a non-empty zero-sum subsequence of length at most 10. Using $D^{10^*}(G) = 12$, we can assume that S is of the following form:

$$S = g_1g_1g_3g_3g_5\cdots g_{15}. \tag{7.7}$$

Define $S' := g_1g_1g_3g_3g_5\cdots g_{13}$ and consider the case $\langle S'\rangle \cong \mathbb{Z}_5^3$. In this case, we deduce that S must have one of the following forms; either

$$S = (1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)g_6\cdots g_{15} \tag{7.8}$$

or

$$S = (1,0,0)(1,0,0)(0,1,0)(0,0,1)g_5g_5g_7\cdots g_{15} \tag{7.9}$$

or

$$S = (1,0,0)(0,1,0)(0,0,1)g_4g_4g_6g_6g_8\cdots g_{15}. \tag{7.10}$$

We build three computer programs, which we name *CPF10U1*, *CPF10U2*, and *CPF10U3*, to generate all sequences over G of length 15 of the three forms above respectively, which do not contain a non-empty zero-sum subsequence of length at most 10. We detail these computer programs in Section 8.2.13. We find that none of the programs produces an output, which is a contradiction. Therefore $D^{10}(G) = 15$.

Very similarly, we show that $D^{11}(G) \leq 14$. Let $S = g_1 \cdots g_{14}$ be an arbitrary sequence over G of length 14. Suppose for a contradiction that S does not contain a non-empty zero-sum subsequence of length at most 11. Using $D^{11^*}(G) = 12$, we can assume that S is of the form (7.7) with g_{15} removed. Define $S' := g_1 g_1 g_3 g_3 g_5 \cdots g_{13}$ and consider the case $\langle S' \rangle \cong \mathbb{Z}_5^3$. In this case S must be of the form (7.8), or (7.9), or (7.10), with g_{15} removed. Similar to before, we build three computer programs, which we name *CPF11U1*, *CPF11U2* , *CPF11U3*, to generate all sequences over G of length 14 of the three forms respectively, which do not contain a non-empty zero-sum subsequence of length at most 11. We detail these computer programs in Section 8.2.14. We find that none of the programs produces an output, which is a contradiction. Therefore $D^{11}(G) = 14$.

We have seen that $14 \leq D^{12}(G)$. Since $D^{12}(G) \leq D^{11}(G) = 14$, we conclude that $D^{12}(G) = 14$.

Let $k \geq 13$ be an integer. It is clear that the sequence over G of length 12 containing each of the elements $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ exactly four times is zero-sum free. From this we deduce that $13 \leq D^k(G)$. Now note that $D^k(G) \leq 13$ as $D(G) = 13$. Hence $D^k(G) = 13$. □

Definition 7.2.10. Let $k \in \mathbb{N}$. For a group G, define $D_k(G)$ to be the smallest $n \in \mathbb{N}$ such that every sequence over G of length n contains k non-empty disjoint zero-sum subsequences.

Remark 7.2.11. Let G be a finite abelian group. Then $D_k(G) \leq kD(G)$ for all $k \in \mathbb{N}$. In particular, if $k = 1$, then $D_k(G) = D(G)$.

Corollary 7.2.12. *Define $G := \mathbb{Z}_5^3$. Then $D_2(\mathbb{Z}_5^3) = 20$.*

Proof. Consider the sequence S over G of length 19 which consists of the elements $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ each repeated four times, the element $(0, 1, 1)$ repeated three times, and the elements $(1, 1, 0)$ and $(1, 0, 1)$ each repeated twice. We build a computer program, which we name *CPF19*, to show that S does not contain a non-empty zero-sum subsequence of length at most 6 or at least 14. We detail CPF19 in Section 8.2.17. We claim that S does not contain two non-empty disjoint zero-sum subsequences. Suppose for a contradiction that S contains two non-empty disjoint zero-sum subsequences S_1 and S_2. It must be that the length of each of these two sequences is at least 7. Hence $S_1 \cup S_2$ is a non-empty zero-sum subsequence of S of length at least 14. This is a contradiction. We deduce that $20 \leq D_2(\mathbb{Z}_5^3)$. Now we claim that $D_2(\mathbb{Z}_5^3) \leq 20$. Let T be an arbitrary sequence over G of length 20. Since $D^7(G) = 19$, we know that T contains a non-empty zero-sum subsequence T_1 of length at most 7. Removing T_1 from T, we obtain a sequence of length at least $13 = D(G)$, which contains a non-empty zero-sum subsequence T_2. It remains to note that T_1 and T_2 are two non-empty disjoint zero-sum subsequences of T. This proves the claim and we deduce that $D_2(\mathbb{Z}_5^3) = 20$. □

Definition 7.2.13. The *Olson constant* $Ol(G)$ of a group G is the smallest $n \in \mathbb{N}$ such that every square-free sequence over G of length n contains a non-empty zero-sum subsequence.

Remark 7.2.14. We have $Ol(G) \leq D(G)$ for all finite abelian groups G.

Corollary 7.2.15. *Define* $G := \mathbb{Z}_5^3$. *Then* $Ol(\mathbb{Z}_5^3) = 12$.

Proof. By the sequence (7.1), we have that $12 \leq Ol(\mathbb{Z}_5^3)$. Now note that $Ol(\mathbb{Z}_5^3) \leq D^{8^*}(G) = 12$. Hence $Ol(\mathbb{Z}_5^3) = 12$. □

7.3 The upper bound

In this chapter we prove the upper bound on $D(\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d})$ that we claimed earlier. More precisely, we prove the following.

Theorem 7.3.1. *Let* $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d}$ *for some* $d \in \mathbb{N}$. *Then*

$$D(G) \leq 1 + d^*(G) + 4.$$

In order to prove this result we shall need the following lemma.

Lemma 7.3.2. *Let $d \in \mathbb{N}$. Then every sequence over $\mathbb{Z}_5^3$ of length $5d + 12$ contains d non-empty disjoint zero-sum subsequences.*

Proof. Let S be a sequence over $\mathbb{Z}_5^3$ of length $5d+12$. We prove the result by induction on d. Suppose $d = 1$. Then the length of S is $17 > 13 = D(\mathbb{Z}_5^3)$. Hence S contains a non-empty zero-sum subsequence. Suppose $d = 2$. Then the length of S is $22 > 20 = D_2(\mathbb{Z}_5^3)$. Hence S contains two non-empty disjoint zero-sum subsequences. Suppose $d = 3$. Then the length of S is 27. Since $D^6(\mathbb{Z}_5^3) = 24$, we can remove a non-empty zero-sum subsequence of length at most 6 from S to obtain a sequence of length at least 21, which contains two non-empty disjoint zero-sum subsequences. Hence we can obtain three non-empty disjoint zero-sum subsequences in S. Suppose $d = 4$. Then the length of S is 32. In this case we can remove two non-empty disjoint zero-sum subsequences of length at most 6 from S to obtain a sequence of length at least 20, which contains two non-empty disjoint zero-sum subsequences. Hence we can obtain four non-empty disjoint zero-sum subsequences in S. Now let $d = k \geq 5$ and suppose the result holds for $d = k - 1$. We have that S is a sequence of length $5k + 12 \geq 37 > 33 = D^5(\mathbb{Z}_5^3)$. Hence S contains a non-empty zero-sum subsequence of length at most 5 which we can remove to obtain at least $5(k - 1) + 12$ elements. By the inductive hypothesis, a sequence of length $5(k - 1) + 12$ over $\mathbb{Z}_5^3$ contains $k - 1$ non-empty disjoint zero-sum subsequences. Hence we can obtain k non-empty disjoint zero-sum subsequences in S. This completes the proof. □

Proof of Theorem 7.3.1. Let S be an arbitrary sequence over G of length $1 + d^*(G) + 4 = 5d + 12$. We claim that S is not zero-sum free. There exists a subgroup H of G such that $H \cong \mathbb{Z}_d$. Define $F := G/H \cong \mathbb{Z}_5^3$. Since $D(H) = d$, by Lemma 4.2.3 it is sufficient to find d disjoint F-zero-sum subsequences in S with respect to the canonical homomorphism $\phi : G \longrightarrow F$ in order to prove the claim. It remains to apply Lemma 7.3.2. □

Chapter 8

Programming searches through sequences over $\mathbb{Z}_3^3$ and $\mathbb{Z}_5^3$

In this chapter we detail the computer programs mentioned in earlier chapters. These programs involve searching for sequences either over $\mathbb{Z}_3^3$ or over $\mathbb{Z}_5^3$ with specific properties.

We split this chapter in two sections. In Section 8.1 we detail the algorithms used to search for specific sequences over $\mathbb{Z}_3^3$. The algorithms used to search for specific sequences over $\mathbb{Z}_5^3$ are detailed in Section 8.2.

8.1 Searches in $\mathbb{Z}_3^3$

In this section we describe the programs mentioned in earlier chapters that search for sequences over $\mathbb{Z}_3^3$ with specific properties. These programs are coded using the computer algebra system Magma and their source codes can be found in the additional files made available with the thesis. When examining the pseudocode in this section, bear in mind that the computer algebra system Magma is equipped with predefined structures such as abelian groups.

8.1.1 CPT9

In this section we describe the computer program CPT9 used in the proof of Lemma 4.3.2. The aim of this program is to generate all sequences over $\mathbb{Z}_3^3$ of length 9 consisting of nine pairwise distinct elements and no non-empty zero-sum subsequence of length at most 3.

The pseudocode for CPT9 can be found in Figure 8.1. Let us explain the algorithm in this pseudocode. The algorithm executes nine for-loops. The first for-loop defines an array of size 1 consisting of an element in $\mathbb{Z}_3^3$. Each subsequent for-loop then defines an array which extends the array created in the preceding for-loop by one element of $\mathbb{Z}_3^3$ subject to an if-condition. The array created in the i-th for-loop represents the sequence of length i over $\mathbb{Z}_3^3$ consisting of the i element(s) determined by the earlier i for-loops.

Now let us explain the if-conditions in the algorithm. Observe that lines 3 and 4 assign a unique number between 1 and 26 to each element in $\mathbb{Z}_3^3\backslash\{0\}$. This means that we can associate an ordered string of numbers to each array as illustrated by the following example: If the numbers associated to the elements $(1, 0, 0)$, $(0, 0, 1)$, $(0, 1, 1)$ are 7, 9 and 4 respectively, then associate the ordered string of numbers 479 to the array $(0, 1, 1)(1, 0, 0)(0, 0, 1)$. The if-condition preceding the creation of an array ensures the ordered string of numbers associated to each array as above consists of pairwise distinct numbers ordered in a strictly increasing fashion. There are two purposes of this. Firstly, the strict inequality between the numbers obtained from the elements in each array ensures the arrays contain pairwise distinct elements of $\mathbb{Z}_3^3$. Secondly, the increasing order of the string of numbers ensures that two arrays do not represent the same sequence over $\mathbb{Z}_3^3$ which makes the algorithm more efficient.

The if-condition on line 18 calls the boolean function hasZeroSum($\bullet$, $\bullet$, $\bullet$) (see Figure 8.2) in order decide whether or not to output the array defined in the 9th for-loop. For an array S, a group G and an integer $n \geq 0$, the function hasZeroSum(S, G, n) returns the boolean value 'true' if and

Pseudocode 1 CPT9

```
define G := Z_3^3
remove the element (0, 0, 0) from G
define an empty list L of size 26
store each element g of G in a unique position L_g in L
for g_1 in L do
    define array S_1 := g_1
    for g_2 in L do
        if L_{g_2} > L_{g_1} then
            define array S_2 := S_1 ∪ g_2
            for g_3 in L do
                if L_{g_3} > L_{g_2} then
                    define array S_3 := S_2 ∪ g_3
                    for g_4 in L do
                            ⋮
                        for g_9 in L do
                            if L_{g_9} > L_{g_8} then
                                define array S_9 := S_8 ∪ g_9
                                if hasZeroSum(S_9, G, 2) is false and
                                   hasZeroSum(S_9, G, 3) is false then
                                    output S_9
                                end if
                            end if
                        end for
                            ⋮
                    end for
                end if
            end for
        end if
    end for
end for
```

Figure 8.1: CPT9

Pseudocode 2 hasZeroSum(S, G, n)

input an array S of size l, a group G, an integer $n \geq 0$
define $I := \{1, \ldots, l\}$
define I' to be the set of all subsets of I of length n
for J in I' **do**
    if calculateValue(extractSubsequence(S, J), G) equals 0_G **then**
        return true
    end if
end for
return false

Figure 8.2: hasZeroSum(S, G, n)

only if the sequence over G represented by the array S contains a zero-sum subsequence of length n. Hence, the calls to this function for each $n \in \{2, 3\}$ with $G = \mathbb{Z}_3^3$ and S as the array defined in the 9th for-loop mean that the algorithm outputs all sequences over $\mathbb{Z}_3^3$ of length 9 consisting of nine pairwise distinct elements and no non-empty zero-sum subsequence of length at most 3.

The pseudocode for the function hasZeroSum($\bullet$, $\bullet$, $\bullet$) can be found in Figure 8.2. Given an array S, a group G and an integer $n \geq 0$, the purpose of the function hasZeroSum(S, G, n) is to check whether or not the sequence over G represented by the array S contains a zero-sum subsequence of length n. The algorithm of this function uses the following naive methodology. Let l denote the size of the array S. For each subset J of size n of the set $\{1, \ldots, l\}$, the algorithm extracts the subarray of S whose elements are indexed by J, computes its value as a sequence over G and checks whether or not the value is equal to 0_G. If a subset of size n of the set $\{1, \ldots, l\}$ is found which indexes a subarray of S whose value is equal to 0_G when considered as a sequence over G then the algorithm returns the boolean value ‘true’. If no such subset of size n of $\{1, \ldots, l\}$ is found the algorithm returns the boolean value 'false'. Hence the algorithm returns the boolean value 'true' if and only if the sequence over G represented by the array S contains a zero-sum subsequence of length n.

Given an array S and a set J, the function extractSubsequence(S, J) described in Figure 8.3 is used to extract the subarray of S indexed by J. The pseudocode for this function is self-explanatory.

Pseudocode 3 extractSubsequence(S, J)

input an array $S = s_1 \cdots s_l$ and a subset J of $\{1, \ldots, l\}$
define an empty array S'
for j in J **do**
    set $S' = S' \cup s_j$
end for
return S'

Figure 8.3: extractSubsequence(S, J)

Given an array S and a group G, the function calculateValue(S, G) described in Figure 8.4 returns the value of S considered as a sequence over G. The pseudocode for this function is also self-explanatory.

Pseudocode 4 calculateValue(S, G)

input an array S and group G
define $v := 0_G$
for s in S **do**
    set $v = v + s$
end for
return v

Figure 8.4: calculateValue(S, G)

8.1.2 CPT10

In this section we describe the computer program CPT10 used in the proof of Lemma 4.3.3. The aim of this program is to generate all sequences over $\mathbb{Z}_3^3$ of length 10 which do not contain a zero-sum subsequence of length $l \in \{1, 2, 3, 8, 9, 10\}$ and which contain the subsequence

$$(1, 0, 0)(1, 0, 0)(0, 1, 0)(0, 1, 0)(0, 0, 1).$$

The algorithm for CPT10 is similar to that of CPT9. We discuss the similarities and differences between the algorithms here. The first four steps of the algorithm for CPT10 consist of executing the steps on the first four lines of Pseudocode 1. The next step in the algorithm for CPT10 is to define an *initial* array

$$S := (1, 0, 0)(1, 0, 0)(0, 1, 0)(0, 1, 0)(0, 0, 1).$$

After this the algorithm executes 5 for-loops. The first for-loop is as follows:

Pseudocode 5 CPT10

$\quad\vdots$
for g_6 in L **do**
  define array $S_6 = S \cup g_6$
  if hasZeroSum(S_6, G, 2) is **false and** hasZeroSum(S_6, G, 3) is **false then**
    for g_7 in L **do**
      $\vdots$
    end for
  end if
end for

These next 4 for-loops behave in a similar fashion to the last 8 for-loops in the algorithm for CPT9 except the following three differences. Firstly, the if-condition preceding the creation of an array in each for-loop contains the inequality $\geq$ instead of $>$. Secondly, each for-loop is directly preceded by an if-condition which ensures the most recently created array does not contain a non-empty zero-sum subsequence of length at most 3 when viewed as a sequence over $\mathbb{Z}_3^3$ (for example see line 4 in Pseudocode 5). The purpose of this if-condition is to speed up the algorithm. Lastly, the if-condition directly before the output command in the last for-loop ensures the array which is output does not contain a zero-sum subsequence of length $l \in \{1, 2, 3, 8, 9, 10\}$ when viewed as a sequence over $\mathbb{Z}_3^3$.

8.1.3 CPT10CNTR

In this section we describe the computer program CPT10CNTR used in the proof of Lemma 4.3.3. Given a sequence $X = x_1 \cdots x_{10}$ over $\mathbb{Z}_3^3$ generated by CPT10, the aims of CPT10CNTR are as follows. Firstly, for each non-empty zero-sum subsequence Y of X, CPT10CNTR computes a $(1, 11)$ matrix X_Y whose $(1, j)$th entry is

$$\begin{cases} 1 & \text{if } x_j \text{ occurs in } Y, \\ -1 & \text{if } j = 11, \\ 0 & \text{otherwise.} \end{cases}$$

Secondly, CPT10CNTR vertically concatenates all matrices X_Y to produce a $z_X \times 11$ matrix A_X where z_X is defined to be the total number of non-empty zero-sum subsequences of X. Thirdly, CPT10CNTR decides whether or not the row Hermite normal of the matrix A_X contains a row of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The pseudocode for CPT10CNTR is presented in Figure 8.5. Given the aims of CPT10CNTR as above, the pseudocode is self-explanatory.

Pseudocode 6 CPT10CNTR

```
define G := Z_3^3
input an array X from the output of CPT10
define c := false
define A_X to be a 0 × 11 matrix
for n ∈ {4, 5, 6, 7} do
    define Z :=zeroSumSubsequencesIndices(X, G, n)
    for Y ∈ Z do
        define X_Y to be the 1 × 11 zero matrix
        set X_{Y_{1,11}} = −1
        for y ∈ Y do
            set X_{Y_{1,y}} = 1
        end for
        Add X_Y as a row of A_X
    end for
end for
put A_X in row Hermite normal form
remove all zero rows from A_X
define r to be the numbers of rows in A_X
if A_{X_{r,1}} = 0 and ··· and A_{X_{r,10}} = 0 and A_{X_{r,11}} = 1 then
    set c = true
end if
return c
```

Figure 8.5: CPT10CNTR

Pseudocode 7 zeroSumSubsequencesIndices(X, G, n)

```
input an array X of size l, a group G, an integer n ≥ 0
define I := {1, . . . , l}
define I′ to be set of all subsets of I of length n
define C = ∅
for J ∈ I′ do
    if calculateValue(extractSubsequence(X, J), G) equals 0_G then
        include J in C
    end if
end for
return C
```

Figure 8.6: zeroSumSubsequencesIndices(X, G, n)

8.1.4 CPT10F

In this section we outline the computer program CPT10F used in the proof of Lemma 4.3.6. The aim of this program is to generate all sequences over $\mathbb{Z}_3^3$ of length 10 containing $(0,1,0)(0,0,1)(0,1,1)$ as a subsequence and no non-empty zero-sum subsequence of length at most 4.

The algorithm for CPT10F is the same as the algorithm for CPT9 with 7 for-loops except the if-condition preceding the creation of an array in each for-loop which contains the inequality $\geq$ instead of $>$, the first and last for-loops, and an extra command defining an initial array as follows:

Pseudocode 8 CPT10F

```
    ⋮
define array S := (0,1,0)(0,0,1)(0,1,1)
for g_4 in L do
    define array S_4 := S ∪ g_4
    for g_5 in L do
            ⋮
        for g_10 in L do
            if L_{g_10} ≥ L_{g_9} then
                define array S_10 := S_9 ∪ g_10
                if hasZeroSum(S_10, G, 2) is false and
                   hasZeroSum(S_10, G, 3) is false and
                   hasZeroSum(S_10, G, 4) is false then
                    output S_10
                end if
            end if
        end for
            ⋮
    end for
end for
```

8.1.5 CPT13

In this section we outline the computer program CPT13 used in the proof of Lemma 4.3.8. The aim of this program is to generate all sequences over $\mathbb{Z}_3^3$ of length 13 without a non-empty zero-sum subsequence of length at most 3 containing the subsequence

$$(0,1,0)(0,1,0)(0,0,1)(0,0,1)(0,1,1)(0,1,1)(1,0,0)(1,0,0). \tag{8.1}$$

The algorithm for CPT13 is the same as the algorithm for CPT10 with the initial array (8.1) except that the if-condition directly before the output command in the last for-loop only ensures the array which is output does not contain a non-empty zero-sum subsequence of length at most 3 when viewed as a sequence over $\mathbb{Z}_3^3$.

8.1.6 CPT13CNTR

In this section we describe the computer program CPT13CNTR used in the proof of Lemma 4.3.8. Given a sequence $X = x_1 \cdots x_{13}$ over $\mathbb{Z}_3^3$ generated by CPT13, the aims of CPT13CNTR are similar to the aims of CPT10CNTR as follows. Firstly, for each non-empty zero-sum subsequence Y of X of length at most 6, CPT13CNTR computes a $(1, 14)$ matrix X_Y whose $(1, j)$th entry is

$$\begin{cases} 1 & \text{if } x_j \text{ occurs in } Y, \\ -1 & \text{if } j = 14, \\ 0 & \text{otherwise.} \end{cases}$$

Secondly, CPT13CNTR vertically concatenates all matrices X_Y to produce a $z_X \times 14$ matrix A_X where z_X is defined to be the total number of non-empty zero-sum subsequences of X of length at most 6. Thirdly, CPT13CNTR decides whether or not the row Hermite normal form of the matrix A_X contains a row of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is simple to see how the pseudocode for CPT10CNTR can be adapted to program CPT13CNTR.

8.1.7 CPT16

In this section we describe the computer program CPT16 used in the proof of Lemma 4.3.10. The aim of CPT16 is to generate all sequences of 16 non-zero elements over $\mathbb{Z}_3^3$ containing the subsequence

$$(0,1,0)(0,0,1)(0,1,1)(1,0,0), \tag{8.2}$$

which contain no zero-sum subsequence of length 3 and no subsequence from the following list of 19 sequences which are the union of a pair of zero-sum sequences of length 2 over $\mathbb{Z}_3^3$:

$$
\begin{aligned}
&(1,0,0)(2,0,0)(1,0,0)(2,0,0),\ (1,0,0)(2,0,0)(2,2,1)(1,1,2),\\
&(1,0,0)(2,0,0)(0,0,1)(0,0,2),\ (1,0,0)(2,0,0)(1,2,1)(2,1,2),\\
&(1,0,0)(2,0,0)(0,1,0)(0,2,0),\ (0,1,0)(0,2,0)(0,1,0)(0,2,0),\\
&(2,0,1)(1,0,2)(2,0,1)(1,0,2),\ (2,1,0)(1,2,0)(2,1,0)(1,2,0),\\
&(0,0,1)(0,0,2)(0,0,1)(0,0,2),\ (2,1,0)(1,2,0)(0,0,1)(0,0,2),\\
&(0,1,0)(0,2,0)(1,2,1)(2,1,2),\ (1,1,2)(2,2,1)(0,0,1)(0,0,2),\\
&(2,1,1)(1,2,2)(2,1,1)(1,2,2),\ (2,1,1)(1,2,2)(0,1,0)(0,2,0),\\
&(2,2,1)(1,1,2)(2,2,1)(1,1,2),\ (1,2,1)(2,1,2)(1,2,1)(2,1,2),\\
&(2,0,1)(1,0,2)(0,1,0)(0,2,0),\ (2,1,1)(1,2,2)(0,0,1)(0,0,2),\\
&(1,2,0)(2,1,0)(2,0,1)(1,0,2).
\end{aligned}
$$

The algorithm for CPT16 is the same as the algorithm for CPT13 with the initial array (8.2) except the following differences. Firstly, CPT16 contains 12 for-loops instead of 5. Secondly, the if-condition directly preceding each of the last 11 for-loops ensures the most recently created array does not contain a zero-sum subsequence of length exactly 3 when viewed as a sequence over $\mathbb{Z}_3^3$. Lastly, the if-condition directly before the output command in the last for-loop ensures the array which is output does not contain a zero-sum subsequence of length exactly 3 and that it does not contain any of the 19 sequences above as a subsequence, when viewed as a sequence over $\mathbb{Z}_3^3$.

8.2 Searches in $\mathbb{Z}_5^3$

In this section we describe the programs mentioned in earlier chapters that search for sequences over $\mathbb{Z}_5^3$ with specific properties. These programs are coded using Magma as well as the Java programming language and their source codes can be found in the additional files made available with the thesis. We shall detail the Java programs first and then the Magma programs.

Java programs

We represent elements of $\mathbb{Z}_5^3$ as integers between 0 and 124 in these programs. We detail this and describe the methods we use to add and find the inverse of elements in $\mathbb{Z}_5^3$ when given as integers between 0 and 124 in Section 8.2.1. We bundle the methods used to manipulate elements of $\mathbb{Z}_5^3$ as integers in an instance of a *class* also known as an *object* in Java. We also represent a sequence over $\mathbb{Z}_5^3$ as an object in Java. We describe the attributes of the classes associated with each of these objects in Section 8.2.2. The subsequent sections contain the pseudocode for the programs mentioned in earlier chapters that search for sequences over $\mathbb{Z}_5^3$. These programs make use of the algorithms described in Section 8.2.1 and Section 8.2.2.

8.2.1 Representing and manipulating elements in $\mathbb{Z}_5^3$

Representing elements in $\mathbb{Z}_5^3$

We represent each element of $\mathbb{Z}_5^3$ as a unique integer between 0 and 124 in the computer programs described in this chapter. More precisely, we represent the element $(a, b, c) \in \mathbb{Z}_5^3$ as $f((a, b, c))$ where f is defined as the following bijective mapping:

$$f : \mathbb{Z}_5^3 \longrightarrow \{0, \ldots, 124\}$$
$$(a, b, c) \mapsto a + 5b + 5^2c.$$

Let us show that f is injective. Let (a, b, c) and (a', b', c') be distinct elements in $\mathbb{Z}_5^3$ and suppose $f((a, b, c)) = f((a', b', c'))$. Then $a+5b+5^2c = a'+5b'+5^2c'$ which implies $a \equiv a' \pmod 5$. Since $a, a' \in [0, 4]$, we obtain that $a = a'$. This means that $b + 5^2c = b' + 5^2c'$ and therefore, applying a similar logic, we deduce that $b = b'$ and $c = c'$. This contradicts the fact that (a, b, c) and (a', b', c') are distinct.

Addition in $\mathbb{Z}_5^3$

The operation of addition on two elements in $\mathbb{Z}_5^3$ represented as integers between 0 and 124 occurs in various places in the computer programs described in the subsequent sections. To carry out this operation efficiently, we use a one-time generated addition table to look up the representation of the sum of any two elements in $\mathbb{Z}_5^3$ as an integer between 0 and 124. In order to generate this table we use a method which takes as input two integers $x, y \in \{0, \ldots, 124\}$, finds $(a, b, c), (a', b', c') \in \mathbb{Z}_5^3$ such that $x = f((a, b, c))$ and $y = f((a', b', c'))$ and outputs the integer $f((a, b, c) + (a', b', c'))$. Before we define this method we recall a definition and make an observation.

Definition 8.2.1. Let a be a non-negative integer. Given $n \in \mathbb{N}$, we define the *base n representation* of a, denoted $a_{(n)}$, to be the string of non-negative integers $a_q a_{q-1} \cdots a_1 a_0$ such that

$$a = a_q n^q + a_{q-1} n^{q-1} + \cdots + a_1 n + a_0,$$

where q is the highest power of n that divides a.

Let $x \in \{0, \ldots, 124\}$ and note that the number of elements in the string $x_{(5)}$ is at most 3. Suppose $x_{(5)} = x_{2-i} \cdots x_0$ for some $i \in \{0, 1, 2\}$. Then we observe that

$$x = \begin{cases} f((x_0, x_1, x_2)) & \text{if } i = 0, \\ f((x_0, x_1, 0)) & \text{if } i = 1, \\ f((x_0, 0, 0)) & \text{if } i = 2. \end{cases}$$

The pseudocode for generating the addition table is described in Figure 8.7.

Pseudocode 9 generateAdditionTable

input $x, y \in \{0, \ldots, 124\}$
compute $x_{(5)}$ and $y_{(5)}$
define two empty lists $l_{x_{(5)}}$, $l_{y_{(5)}}$ of size 3 each
for $\alpha \in \{x_{(5)}, y_{(5)}\}$ **do**
    if number of characters in α equals 1 **then**
        set $l_\alpha = (\alpha, 0, 0)$
    end if
    if number of characters in α equals 2 **then**
        set $l_\alpha = (\alpha_0, \alpha_1, 0)$ where $\alpha = \alpha_1\alpha_0$
    end if
    if number of characters in α equals 3 **then**
        set $l_\alpha = (\alpha_0, \alpha_1, \alpha_2)$ where $\alpha = \alpha_2\alpha_1\alpha_0$
    end if
end for
add $l_{x_{(5)}}$ and $l_{y_{(5)}}$ component-wise and denote the result by $l_{x_{(5)}} + l_{y_{(5)}}$
reduce each component of $l_{x_{(5)}} + l_{y_{(5)}}$ to the least non-negative residue modulo 5
return $a + 5b + 5^2c$ where $l_{x_{(5)}} + l_{y_{(5)}} = (a, b, c)$

Figure 8.7: generateAdditionTable

Given $x, y \in \{0, \ldots, 124\}$, we shall use the notation *additionTable(x, y)* to refer to the integer $f((a, b, c) + (a', b', c'))$ where $x = f((a, b, c))$ and $y = f((a', b', c'))$.

Computing inverses in $\mathbb{Z}_5^3$

The operation of finding the additive inverse of an element in $\mathbb{Z}_5^3$ occurs very frequently in the computer programs we use to search for sequences over $\mathbb{Z}_5^3$. In order to carry out this operation efficiently we retrieve the inverse of an element in $\mathbb{Z}_5^3$ from a one-time generated list of size 125 indexed by the integers $0, \ldots, 124$. In this list, the entry indexed by i is the inverse of the element $(a, b, c) \in \mathbb{Z}_5^3$ such that $f((a, b, c)) = i$. The pseudocode for generating this list is described in Figure 8.8.

Pseudocode 10 inverse(i)

```
input an integer i ∈ {0, ..., 124}
for integers j ∈ {0, ..., 124} do
    if additionTable(i, j)=0 then
        return j
    end if
end for
```

Figure 8.8: inverse(i)

8.2.2 Groups and sequences as objects in Java

Each program described in subsequent sections consists of three Java classes. One of these classes is the class which contains the *main* method. The pseudocode of this method is what is described in the subsequent sections for each program. The other two classes are *Group.java* and *GroupSequence.java* as described below.

Group.java

We implement a group in the Java programming language as a class called Group.java. We create $\mathbb{Z}_5^3$ as an instance of this class. All the methods described in Section 8.2.1 are placed in Group.java as can be seen in the source codes of the programs.

GroupSequence.java

We implement a sequence in the Java programming language as a class called GroupSequence.java. We create a particular sequence as an instance of this class. The class GroupSequence has the following attributes:

- A list L, the purpose of which is to store elements of the sequence.
- An integer l which refers to the length of the sequence.
- A $125 \times l$ grid with boolean entries with rows indexed by $0, \ldots, 124$ and columns indexed by $1, \cdots, l$ in which the (i, j)-th entry is 'true' if and only if the element $(a, b, c) \in \mathbb{Z}_5^3$ such that $i = f((a, b, c))$ satisfies $(a, b, c) = t_1 + \cdots + t_j$ for some sublist $t_1 \cdots t_j$ of L of length j. In plain words, this grid stores the set of subsums of the sequence along with the lengths of the (non-empty) subsequences from which the subsums arise. We shall refer to this grid as the *grid of subsums* of this sequence.

Given a sequence S over $\mathbb{Z}_5^3$ as an instance of the class GroupSequence.java, we shall denote its grid of subsums by G_S and the (i, j)-th entry of G_S by $G_S(i, j)$. Each entry in G_S is initialised to 'false' at the time the sequence S is created as an instance of GroupSequence.java.

There are many occurrences in the computer programs that we mention in subsequent sections where, given a sequence S over $\mathbb{Z}_5^3$ and an element $g \in \mathbb{Z}_5^3$, the grid of subsums of the sequence $S \cup g$ needs to be computed. The class GroupSequence.java contains a method called updateSubsums($\bullet$, $\bullet$), as described in Figure 8.9, for this purpose. We only apply this method given that each entry in $G_{S \cup g}$ is initialised to 'false'.

The attributes of the class GroupSeqeunce.java are as described above for all computer programs described in the subsequent sections except CPF6. The differences in this class for CPF6 are described in the Section 8.2.3.

Pseudocode 11 updateSubsums(S, g)

```
input a sequence S of length l and an element g ∈ Z_5^3
set G_{S∪g}(f(g), 1) to true
for integers i ∈ {0, ..., 124} do
    for integers j ∈ {1, ..., l} do
        if G_S(i, j) is true then
            set G_{S∪g}(i, j) to true
            set G_{S∪g}(additionTable(i, f(g)), j + 1) to true
        end if
    end for
end for
```

Figure 8.9: updateSubsums(S, g)

8.2.3 CPF6

In this section we describe the computer program CPF6 used in the proof of Theorem 6.1. The aim of CPF6 is to generate all zero-sum free sequences over $\mathbb{Z}_5^3$ of length 6 containing the subsequence $(1, 0, 0)(0, 1, 0)(0, 0, 1)$.

The class GroupSequence.java differs slightly for CPF6 from as it is described previously in the sense that instead of a grid, we define a 125×1 list G_S with boolean entries indexed by $0, \ldots, 124$, each time a sequence S is created as an instance of the class. The i-th entry, $G_S(i)$, in this list is 'true' if and only if the element $(a, b, c) \in \mathbb{Z}_5^3$ such that $i = f((a, b, c))$ satisfies $(a, b, c) = t_1 + \cdots + t_j$ for some subsequence $t_1 \cdots t_j$ of S of some length $j \neq 0$. The method updateSubsums($\bullet$, $\bullet$) is defined similarly. The main reason for this difference is simply that the storage of lengths of the subsequences from which the subsums arise is not necessary for the purposes of CPF6.

The pseudocode for CPF6 is as described in Figure 8.10. Let us explain the algorithm in the pseudocode for CPTF6. We start with the initial (zero-sum free) sequence $(1, 0, 0)(0, 1, 0)(0, 0, 1)$ and aim to sequentially extend it by three more elements in $\mathbb{Z}_5^3$ whilst making sure the resulting sequence at each step is zero-sum free. Each of these steps is executed by means of a for-loop. Each for-loop contains an if-condition the purpose of which is to ensure the sequence created in that for-loop is zero-sum free. More precisely, the if-condition directly following the start of a for-loop ensures the inverse of the element determined by that for-loop does not occur as a subsum of the most recently created sequence. The following statement aids the explanation of the if-condition: 'Given a zero-sum free sequence S over a group G and an element $g \in S$, the sequence $S \cup g$ is zero-sum free if and only if $-g \notin [S]$.' The purpose of the inequalities in the if-conditions is to ensure we do not generate the same sequence more than once. The remaining pseudocode is self-explanatory.

8.2.4 CPF14

In this section we describe the computer program CPF14 used in the proof of Theorem 6.1. The aim of CPF14 is to take a possibility U generated by CPF6 as input and output all sequences X over $\mathbb{Z}_5^3$ of length 14 subject to the following conditions:

- The sequence X contains U as a subsequence;
- The sequence X does not contain a non-empty zero-sum subsequence of even length or length at most 5.

The pseudocode for CPF14 is as described in Figure 8.11. Let us explain the algorithm in the pseudocode for CPF14. We start with a sequence U output from CPF6. We ensure that G_U is updated to reflect the subsums of U along with the lengths of the subsequences from which they arise. The aim of the algorithm is to sequentially extend U by 8 more elements in $\mathbb{Z}_5^3$ whilst making sure the resulting sequence at each step does not contain a non-empty zero-sum subsequence of even length or length at most 5. Similar

Pseudocode 12 CPF6

```
create sequence S = (1,0,0)(0,1,0)(0,0,1)
set G_S(i) = true for all i ∈ {1,5,6,25,26,30,31}
for g_4 ∈ Z_5^3 \ {0} do
    if G_S(f(−g_4)) is false then
        create sequence S_4 = S ∪ g_4
        call updateSubsums(S, g_4)
        for g_5 ∈ Z_5^3 \ {0} do
            if f(g_5) ≥ f(g_4) and G_{S_4}(f(−g_5)) is false then
                create sequence S_5 = S_4 ∪ g_5
                call updateSubsums(S_4, g_5)
                for g_6 ∈ Z_5^3 \ {0} do
                    if f(g_6) ≥ f(g_5) and G_{S_5}(f(−g_6)) is false then
                        create sequence S_6 = S_5 ∪ g_6
                        call updateSubsums(S_5, g_6)
                        output S_6
                    end if
                end for
            end if
        end for
    end if
end for
```

Figure 8.10: CPF6

to the algorithm for CPF6, we use for-loops with if-conditions to achieve this aim. The if-condition directly following the start of a for-loop ensures the inverse of the element determined by that for-loop does not occur as a subsum of length $j \in \{1,2,3,4,5,7,9,11,13\}$ of the most recently created sequence. This ensures the sequence created in that for-loop does not contain a non-empty zero-sum subsequence of even length or length at most 5. We conclude the explanation for the pseudocode of CPF14 here as the remaining is either similar to that of CPF6 or self-explanatory.

Pseudocode 13 CPF14

input a sequence U output from CPF6 along with G_U
for $g_7 \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
  if $G_U(f(-g_7), j)$ is **false** for all $j \in \{1, 2, 3, 4, 5\}$ **then**
    create sequence $S_7 = U \cup g_7$
    call updateSubsums(U, g_7)
    for $g_8 \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
      if $f(g_8) \geq f(g_7)$ **and** $G_{S_7}(f(-g_8), j)$ is **false** for all $j \in \{1, 2, 3, 4, 5, 7\}$ **then**
        create sequence $S_8 = S_7 \cup g_8$
        call updateSubsums(S_7, g_8)
        for $g_9 \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
          if $f(g_9) \geq f(g_8)$ **and** $G_{S_8}(f(-g_9), j)$ is **false** for all $j \in \{1, 2, 3, 4, 5, 7\}$ **then**
            create sequence $S_9 = S_8 \cup g_9$
            call updateSubsums(S_8, g_9)
            for $g_{10} \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
              if $f(g_{10}) \geq f(g_9)$ **and** $G_{S_9}(f(-g_{10}), j)$ is **false** for all $j \in \{1, 2, 3, 4, 5, 7, 9\}$ **then**
                create sequence $S_{10} = S_9 \cup g_{10}$
                call updateSubsums(S_9, g_{10})
                for $g_{11} \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
                  if $f(g_{11}) \geq f(g_{10})$ **and** $G_{S_{10}}(f(-g_{11}), j)$ is **false** for all $j \in \{1, 2, 3, 4, 5, 7, 9\}$ **then**
                    create sequence $S_{11} = S_{10} \cup g_{11}$
                    call updateSubsums(S_{10}, g_{11})
                    for $g_{12} \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
                      if $f(g_{12}) \geq f(g_{11})$ **and** $G_{S_{11}}(f(-g_{12}), j)$ is **false** for all $j \in \{1, 2, 3, 4, 5, 7, 9, 11\}$ **then**
                        create sequence $S_{12} = S_{11} \cup g_{12}$
                        call updateSubsums(S_{11}, g_{12})

for $g_{13} \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**

if $f(g_{13}) \geq f(g_{12})$ **and** $G_{S_{12}}(f(-g_{13}), j)$ is **false** for all $j \in \{1, 2, 3, 4, 5, 7, 9, 11\}$ **then**

create sequence $S_{13} = S_{12} \cup g_{13}$

call updateSubsums(S_{12}, g_{13})

for $g_{14} \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**

if $f(g_{14}) \geq f(g_{13})$ **and** $G_{S_{13}}(f(-g_{14}), j)$ is **false** for all $j \in \{1, 2, 3, 4, 5, 7, 9, 11, 13\}$ **then**

create sequence $S_{14} = S_{13} \cup g_{14}$

call updateSubsums(S_{13}, g_{14})

output S_{14}

end if

end for

end if

end for

end if

end for

end if

end for

end if

end for

end if

end for

end if

end for

end if

end for

Figure 8.11: CPF14

8.2.5 CPF14EXT

In this section we describe the computer program CPF14EXT used in the proof of Theorem 6.1. The aim of CPF14EXT is to take as input a possibility X for $x_1 \cdots x_{14}$ generated by CPF14 and output all sequences $X \cup z_{15} \cdots z_{18}$ over $\mathbb{Z}_5^3$ of length 18 subject to the following conditions:

1. The sequence $X \cup z_{15} \cdots z_{18}$ does not contain a non-empty zero-sum subsequence of length at most 5 or length at least 14;
2. The inverse of the value of each of the following sequences does not occur as the value of some subsequence of X of even length: z_{15}, z_{16}, z_{17}, z_{18}, $z_{15}z_{16}$, $z_{15}z_{17}$, $z_{15}z_{18}$, $z_{16}z_{18}$, $z_{17}z_{18}$, $z_{15}z_{16}z_{17}$, $z_{16}z_{17}z_{18}$, $z_{15}z_{16}z_{17}z_{18}$.

The pseudocode for CPF14EXT is as described in Figure 8.12. In this algorithm, the notation $[S]_L$ for a sequence S over $\mathbb{Z}_5^3$ and a set of natural numbers L, denotes the set

$$\{g \in \mathbb{Z}_5^3 \mid G_S(f(g), j) \text{ is 'true' for some } j \in L\}.$$

In simpler terms, the set $[S]_L$ consists precisely of the elements of $\mathbb{Z}_5^3$ which occur as a subsum of S arising from a subsequence of S of length $j \in L$. The structure of the algorithm for CPF14EXT is somewhat similar to the algorithm described previously. It contains four for-loops each of which extends the input sequence X by one element. Each for-loop determines an element of the set $\mathbb{Z}_5^3 \backslash \{0, [X]_{L_1}\}$ where $L_1 := \{1, 2, 3, 4, 6, 8, 10, 12, 13, 14\}$. This reduces the number of candidates for the four elements to be appended to X firstly by ensuring the inverse of each of the four elements appended to X does not occur as the value of some subsequence of X of even length, and secondly by ensuring the sequence to be created in the subsequent for-loop does not contain a non-empty zero-sum subsequence of length at most 5 or 14 of a particular form. The last and-statement in each of the if-conditions directly following each for-loop ensures the output sequence does not contain a non-empty zero-sum subsequence of length at most 5 or length at least 14 of any form. The last if-condition in the algorithm ensures (2) from the start of this subsection is satisfied.

Pseudocode 14 CPF14EXT

input a sequence X output from CPF14 along with G_X
if $\mathbb{Z}_5^3 \setminus \{0, [X]_{L_1}\} \neq \emptyset$ for $L_1 := \{1, 2, 3, 4, 6, 8, 10, 12, 13, 14\}$ **then**
  for $z_{15} \in \mathbb{Z}_5^3 \setminus \{0, [X]_{L_1}\}$ **do**
    create sequence $S_{15} = X \cup -z_{15}$
    call updateSubsums(X, $-z_{15}$)
    if $\mathbb{Z}_5^3 \setminus \{0, [S_{15}]_{L_2}\} \neq \emptyset$ for $L_2 := \{1, 2, 3, 4, 13, 14, 15\}$ **then**
      for $z_{16} \in \mathbb{Z}_5^3 \setminus \{0, [X]_{L_1}\}$ **do**
        if $f(z_{16}) \geq f(z_{15})$ **and** $z_{16} \in \mathbb{Z}_5^3 \setminus \{0, [S_{15}]_{L_2}\}$ **then**
          create sequence $S_{16} = S_{15} \cup -z_{16}$
          call updateSubsums(S_{15}, $-z_{16}$)
          if $\mathbb{Z}_5^3 \setminus \{0, [S_{16}]_{L_3}\} \neq \emptyset$ for $L_3 := \{1, 2, 3, 4, 13, 14, 15, 16\}$ **then**
            for $z_{17} \in \mathbb{Z}_5^3 \setminus \{0, [X]_{L_1}\}$ **do**
              if $f(z_{17}) \geq f(z_{18})$ **and** $z_{17} \in \mathbb{Z}_5^3 \setminus \{0, [S_{15}]_{L_2}\}$ **and** $z_{17} \in \mathbb{Z}_5^3 \setminus \{0, [S_{16}]_{L_3}\}$ **then**
                create sequence $S_{17} = S_{16} \cup -z_{17}$
                call updateSubsums(S_{16}, $-z_{17}$)
                if $\mathbb{Z}_5^3 \setminus \{0, [S_{17}]_{L_4}\} \neq \emptyset$ for $L_4 := \{1, 2, 3, 4, 13, 14, 15, 16, 17\}$ **then**
                  for $z_{18} \in \mathbb{Z}_5^3 \setminus \{0, [X]_{L_1}\}$ **do**
                    if $f(z_{18}) \geq f(z_{17})$ **and** $z_{18} \in \mathbb{Z}_5^3 \setminus \{0, [S_{15}]_{L_2}\}$
                    and $z_{18} \in \mathbb{Z}_5^3 \setminus \{0, [S_{16}]_{L_3}\}$ **and** $z_{18} \in \mathbb{Z}_5^3 \setminus \{0, [S_{17}]_{L_4}\}$ **then**
                    if $g \in \mathbb{Z}_5^3 \setminus [X]_{L_5}$ for all $g \in \{z_{15} + z_{16}, z_{15} + z_{17}, z_{15} + z_{18}, z_{16}+z_{18}, z_{17}+z_{18}, z_{15}+z_{16}+z_{17}, z_{16}+z_{17}+z_{18}, z_{15}+z_{16}+z_{17}+z_{18}\}$
                    where $L_5 := \{2, 4, 6, 8, 10, 12, 14\}$ **then**
                      output $S_{17}, -z_{18}$
                    end if

```
                    end if
                  end for
                end if
              end if
            end for
          end if
        end if
      end for
    end if
  end for
end if
```

Figure 8.12: CPF14EXT

8.2.6 CPF12

In this section we describe the computer program CPF12 used in the proof of Theorem 6.1. The aim of CPF12 is to take a possibility U generated by CPF6 as input and output all sequences X over $\mathbb{Z}_5^3$ of length 12 subject to the following conditions:

- The sequence X contains U as a subsequence;
- The sequence X does not contain a non-empty zero-sum subsequence of even length or length at most 5.

The algorithm used for CPF12 is the same as the algorithm for CPF14 with the obvious exception that we use 6 for-loops instead of 8.

8.2.7 CPF12EXT

In this section we describe the computer program CPF12EXT used in the proof of Theorem 6.1. The aim of CPF12EXT is to take as input a possibility X for $x_1 \cdots x_{12}$ generated by CPF12 and output all sequences $X \cup z_{13} \cdots z_{18}$ over $\mathbb{Z}_5^3$ of length 18 subject to the following conditions:

1. The sequence $X \cup z_{13} \cdots z_{18}$ does not contain a non-empty zero-sum subsequence of length at most 5 or length at least 14;
2. The inverse of the value of each subsequence of $z_{13} \cdots z_{18}$ does not occur as the value of some subsequence of X of even length.

The algorithm used for CPF12EXT is the same as the algorithm for CPF14EXT with the exception that it contains 6 for-loops instead of 4 and that the last if-condtion is modified to reflect (2) above.

8.2.8 CPF5U*, CPF6U*, CPF7U* and CPF8U*

In this section we describe the computer programs CPF5U*, CPF6U*, CPF7U* and CPF8U* used in the proof of Theorem 7.2.9.

The aim of CPF5U* is to generate all square-free sequences over $\mathbb{Z}_5^3$ of length 15 containing $(1,0,0)(0,1,0)(0,0,1)$ as a subsequence and no non-empty zero-sum subsequence of length at most 5. The algorithm for CPF5U* is similar to that of the program CPF14. We present the pseudocode of CPF5U* in Figure 8.13 without further explanation.

The aim of CPF6U*, respectively CPF7U*, is to generate all square-free sequences over $\mathbb{Z}_5^3$ of length 14, respectively 13, containing $(1,0,0)(0,1,0)(0,0,1)$ as a subsequence and no non-empty zero-sum subsequence of length at most 6, respectively at most 7. The algorithms for CPF6U* and CPF7U* are analogous to that of the program CPF5U* with the set $[1,4]$ replaced with $[1,5]$, respectively $[1,6]$, in each of the if-conditions.

The aim of CPF8U* is to generate all square-free sequences over $\mathbb{Z}_5^3$ of length 12 containing $(1,0,0)(0,1,0)$ as a subsequence and no non-empty zero-sum subsequence of length at most 8. The algorithm for CPF8U* is also similar to that of CPF5U* with the set $[1,4]$ replaced with $[1,7]$ in each of the if-conditions.

Pseudocode 15 CPF5U*

```
create sequence S_1 = (1,0,0)
set G_{S_1}(1,1) to true
create sequence S_2 = S_1 ∪ (0,1,0)
call updateSubsums(S_1, (0,1,0))
create sequence S_3 = S_2 ∪ (0,0,1)
call updateSubsums(S_2, (0,0,1))
for g_4 ∈ Z_5^3 \ {0, (1,0,0), (0,1,0), (0,0,1)} do
    if G_{S_3}(f(−g_4), j) is false for all j ∈ [1,4] then
        create sequence S_4 = S_3 ∪ g_4
        call updateSubsums(S_3, g_4)
        for g_5 ∈ Z_5^3 \ {0, (1,0,0), (0,1,0), (0,0,1)} do
            if f(g_5) > f(g_4) and G_{S_4}(f(−g_5), j) is false for all j ∈ [1,4] then
                create sequence S_5 = S_4 ∪ g_5
                call updateSubsums(S_4, g_5)
                    ⋮
                for g_15 ∈ Z_5^3 \ {0, (1,0,0), (0,1,0), (0,0,1)} do
                    if f(g_15) > f(g_14) and G_{S_14}(f(−g_15), j) is false for all j ∈ [1,4] then
                        create sequence S_15 = S_14 ∪ g_15
                        output S_15
                    end if
                end for
            end if
        end for
    end if
end for
```

Figure 8.13: CPF5U*

8.2.9 CPF6U1 and CPF6U2

In this section we describe the computer programs CPF6U1 and CPF6U2 used in the proof of Theorem 7.2.9.

The aim of CPF6U1 is to generate all sequences over $\mathbb{Z}_5^3$ of the form

$$(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)(0,0,1)g_7g_7g_9g_9g_{11}g_{11}g_{13}\cdots g_{24},$$

which do not contain a non-empty zero-sum subsequence of length at most 6. We present the pseudocode of CPF6U1 in Figure 8.14.

The aim of CPF6U2 is to generate all sequences over $\mathbb{Z}_5^3$ of the form

$$(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)g_6g_6g_8g_8g_{10}g_{10}g_{12}g_{12}g_{14}\cdots g_{24},$$

which do not contain a non-empty zero-sum subsequence of length at most 6. The algorithm for CPF6U2 is similar to that of CPF6U1.

Pseudocode 16 CPF6U1

create sequence $S_1 = (1,0,0)$
set $G_{S_1}(1,1)$ to **true**
create sequence $S_2 = S_1 \cup (1,0,0)$
call updateSubsums(S_1, $(1,0,0)$)
create sequence $S_3 = S_2 \cup (0,1,0)$
call updateSubsums(S_2, $(0,1,0)$)
create sequence $S_4 = S_3 \cup (0,1,0)$
call updateSubsums(S_3, $(0,1,0)$)
create sequence $S_5 = S_4 \cup (0,0,1)$
call updateSubsums(S_4, $(0,0,1)$)
create sequence $S_6 = S_5 \cup (0,0,1)$
call updateSubsums(S_5, $(0,0,1)$)
for $g_7 \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
    if $G_{S_6}(f(-g_7), j)$ is **false** for all $j \in [1,5]$ **then**
        create sequence $S_7 = S_6 \cup g_7$
        call updateSubsums(S_6, g_7)
        if $G_{S_7}(f(-g_7), j)$ is **false** for all $j \in [1,5]$ **then**
            create sequence $S_8 = S_7 \cup g_7$
            call updateSubsums(S_7, g_7)
            for $g_9 \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
                if $g_9 \geq g_7$ **and** $G_{S_8}(f(-g_9), j)$ is **false** for all $j \in [1,5]$ **then**
                    create sequence $S_9 = S_8 \cup g_9$
                    call updateSubsums(S_8, g_9)
                    if $G_{S_9}(f(-g_9), j)$ is **false** for all $j \in [1,5]$ **then**
                        create sequence $S_{10} = S_9 \cup g_9$
                        call updateSubsums(S_9, g_9)

for $g_{11} \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
  if $g_{11} \geq g_9$ **and** $G_{S_{10}}(f(-g_{11}), j)$ is **false** for all $j \in [1,5]$ **then**
    create sequence $S_{11} = S_{10} \cup g_{11}$
    call updateSubsums(S_{10}, g_{11})
    if $G_{S_{11}}(f(-g_{11}), j)$ is **false** for all $j \in [1,5]$ **then**
      create sequence $S_{12} = S_{11} \cup g_{11}$
      call updateSubsums(S_{11}, g_{g11})
      for $g_{13} \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
        if $G_{S_{12}}(f(-g_{13}), j)$ is **false** for all $j \in [1,5]$ **then**
          create sequence $S_{13} = S_{12} \cup g_{13}$
          call updateSubsums(S_{12}, g_{g13})
          for $g_{14} \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
            if $g_{14} \geq g_{13}$ **and** $G_{S_{13}}(f(-g_{14}), j)$ is **false** for all $j \in [1,5]$ **then**
              create sequence $S_{14} = S_{13} \cup g_{14}$
              call updateSubsums(S_{13}, g_{g14})
                $\vdots$
              for $g_{24} \in \mathbb{Z}_5^3 \setminus \{0\}$ **do**
                if $g_{24} \geq g_{23}$ **and** $G_{S_{23}}(f(-g_{24}), j)$ is **false** for all $j \in [1,5]$ **then**
                  create sequence $S_{24} = S_{23} \cup g_{24}$
                  output S_{24}

```
                            end if
                          end for
                        end if
                      end for
                    end if
                  end for
                end if
              end if
            end for
          end if
        end if
      end for
    end if
  end if
end for
```

Figure 8.14: CPF6U1

8.2.10 CPF7U1, CPF7U2, CPF7U3 and CPF7U4

In this section we describe the computer programs CPF7U1, CPF7U2, CPF7U3 and CPF7U4 used in the proof of Theorem 7.2.9.

The respective aim of CPF7U1, CPF7U2, CPF7U3 and CPF7U4 is to generate all sequences over $\mathbb{Z}_5^3$ of the four forms below, each of which do not contain a non-empty zero-sum subsequence of length at most 7:

$$(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)(0,0,1)g_7g_7g_9\cdots g_{19},$$

$$(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)g_6g_6g_8g_8g_{10}\cdots g_{19},$$

$$(1,0,0)(1,0,0)(0,1,0)(0,0,1)g_5g_5g_7g_7g_9g_9g_{11}\cdots g_{19},$$

$$(1,0,0)(0,1,0)(0,0,1)g_4g_4g_6g_6g_8g_8g_{10}g_{10}g_{12}\cdots g_{19}.$$

The algorithm for these programs is similar to that of CPF6U1 with the set $[1,5]$ replaced with $[1,6]$.

8.2.11 CPF8U1, CPF8U2, CPF8U3 and CPF8U4

In this section we describe the computer programs CPF8U1, CPF8U2, CPF8U3 and CPF8U4 used in the proof of Theorem 7.2.9.

The respective aim of CPF8U1, CPF8U2, CPF8U3 and CPF8U4 is to generate all sequences over $\mathbb{Z}_5^3$ of the four forms in Section 8.2.10 with g_{19} removed, each of which do not contain a non-empty zero-sum subsequence of length at most 8. It is easy to see how the algorithm for these programs can be adapted from the algorithm for the programs in Section 8.2.10

8.2.12 CPF9U1, CPF9U2, CPF9U3 and CPF9U4

In this section we describe the computer programs CPF9U1, CPF9U2, CPF9U3 and CPF9U4 used in the proof of Theorem 7.2.9.

The respective aim of CPF9U1, CPF9U2, CPF9U3 and CPF9U4 is to generate all sequences S over $\mathbb{Z}_5^3$ of the four forms below, each of which do not contain a non-empty zero-sum subsequence of length at most 9:

$$(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)(0,0,1)g_7\cdots g_{17},$$

$$(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)g_6g_6g_8\cdots g_{17},$$

$$(1,0,0)(1,0,0)(0,1,0)(0,0,1)g_5g_5g_7g_7g_9\cdots g_{17},$$

$$(1,0,0)(0,1,0)(0,0,1)g_4g_4g_6g_6g_8g_8g_{10}\cdots g_{17}.$$

The algorithm for these programs is similar to that of CPF6U1 with the set $[1,5]$ replaced with $[1,8]$.

8.2.13 CPF10U1, CPF10U2 and CPF10U3

In this section we describe the computer programs CPF10U1, CPF10U2 and CPF10U3 used in the proof of Theorem 7.2.9.

The respective aim of CPF10U1, CPF10U2 and CPF10U3 is to generate all sequences over $\mathbb{Z}_5^3$ of the three forms below, each of which do not contain a non-empty zero-sum subsequence of length at most 10:

$$(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,0,1)g_6\cdots g_{15},$$

$$(1,0,0)(1,0,0)(0,1,0)(0,0,1)g_5g_5g_7\cdots g_{15},$$

$$(1,0,0)(0,1,0)(0,0,1)g_4g_4g_6g_6g_8\cdots g_{15}.$$

The algorithm for these programs is similar to that of CPF6U1 with the set $[1,5]$ replaced with $[1,9]$.

8.2.14 CPF11U1, CPF11U2 and CPF11U3

In this section we describe the computer programs CPF11U1, CPF11U2 and CPF11U3 used in the proof of Theorem 7.2.9.

The respective aim of CPF11U1, CPF11U2 and CPF11U3 is to generate all sequences over $\mathbb{Z}_5^3$ of the three forms in Section 8.2.13 with g_{15} removed, each of which do not contain a non-empty zero-sum subsequence of length at most 11. It is easy to see how the algorithm for these programs can be adapted from the algorithm for the programs in Section 8.2.13.

Magma programs

8.2.15 CPF5L*, CPF6L*, CPF7L* and CPF8L*

In this section we describe the computer programs CPF5L*, CPF6L*, CPF7L* and CPF8L* used in the proof of Theorem 7.2.9.

The respective aim of CPF5L*, CPF6L*, CPF7L* and CPF8L* is to verify that the following four sequences over $\mathbb{Z}_5^3$ do not contain a non-empty zero-sum subsequence of length at most 5, at most 6, at most 7, and at most 11 respectively:

$$\begin{aligned}&(1,0,0)(0,1,0)(0,0,1)(2,0,0)(1,1,0)(2,1,0)(3,1,0)(1,0,1)\\&(2,0,1)(3,0,1)(0,1,1)(1,1,1)(2,1,1)(3,1,1),\end{aligned}\tag{8.3}$$

Pseudocode 17 CPF5L*

```
define G := Z_5^3
define array S = (8.3)
if hasZeroSum(S, G, 1) is false and hasZeroSum(S, G, 2) is false and
    hasZeroSum(S, G, 3) is false and hasZeroSum(S, G, 4) is false and
    hasZeroSum(S, G, 5) is false then
    output true
end if
```

Figure 8.15: CPF5L*

$$(1,0,0)(0,1,0)(0,0,1)(2,0,0)(1,1,0)(2,1,0)(3,1,0)(1,0,1)$$
$$(2,0,1)(3,0,1)(0,1,1)(1,1,1)(2,1,1),$$

$$(1,0,0)(0,1,0)(0,0,1)(2,0,0)(1,1,0)(2,1,0)(3,1,0)(1,0,1)$$
$$(2,0,1)(3,0,1)(0,1,1)(1,1,1),$$

$$(1,0,0)(0,1,0)(0,0,1)(2,0,0)(1,1,0)(1,0,1)(2,1,0)(3,1,0)$$
$$(2,0,1)(3,0,1)(4,1,1).$$

All of these programs follow a similar algorithm. Therefore, we only detail the pseudocode for CPF5L*, which can be found in Figure 8.15.

8.2.16 CPF6L, CPF7L, CPF8L, CPF9L, CPF10L and CPF12L

In this section we describe the computer programs CPF6L, CPF7L, CPF8L, CPF9L, CPF10L and CPF12L used in the proof of Theorem 7.2.9.

The respective aim of CPF6L, CPF7L, CPF8L, CPF9L, CPF10L and

CPF12L is to verify that the following six sequences over $\mathbb{Z}_5^3$ do not contain a non-empty zero-sum subsequence of length at most 6, at most 7, at most 8, at most 9, at most 10 and at most 12 respectively:

$$(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$$
$$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(4,1,1)(4,1,1)(4,1,1)(4,1,1)$$
$$(1,1,0)(1,1,0)(1,1,0)(1,0,1)(1,0,1)(1,0,1)(0,1,1),$$

$$(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$$
$$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(4,1,1)(4,1,1)(4,1,1)(4,1,1)$$
$$(1,1,0)(1,1,0),$$

$$(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$$
$$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(4,1,1)(4,1,1)(4,1,1)(4,1,1)$$
$$(1,1,0),$$

$$(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$$
$$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(4,1,1)(4,1,1)(4,1,1)(4,1,1),$$

$$(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$$
$$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(1,1,1)(1,1,1),$$

$$(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$$
$$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(1,1,1).$$

All of these programs follow a similar algorithm to CPF5L*.

8.2.17 CPF19

In this section we describe the computer program CPF19 used in the proof of Corollary 7.2.12. The aim of CPF19 is to verify that the following sequence over $\mathbb{Z}_5^3$ does not contain a non-empty zero-sum subsequence of length at most 6 or at least 14:

$$(1,0,0)(1,0,0)(1,0,0)(1,0,0)(0,1,0)(0,1,0)(0,1,0)(0,1,0)$$
$$(0,0,1)(0,0,1)(0,0,1)(0,0,1)(0,1,1)(0,1,1)(0,1,1)(1,1,0)$$
$$(1,1,0)(1,0,1)(1,0,1).$$

It is easy to see how the algorithm for CPF5L* can be adapted for CPF19.

Chapter 9

Upper bounds on $D(G)$ in terms of $d^*(G)$

9.1 Motivation

Fix a group G and define $d := d^*(G)$. Recall the following trivial lower bound on $D(G)$ from Lemma 1.2.2:

$$D(G) \geq 1 + d.$$

This trivial lower bound motivates the following question: what is the best upper bound on $D(G)$ in terms of d? We address this question in this chapter. Surprisingly, if G is not listed in List 2.1.7 then no such upper bound has previously been proved, to the author's knowledge. Using a result of P. van Emde Boas and D. Kruyswijk, we find the following general polynomial upper bound on $D(G)$ in terms of d when G is not a p-group and $\mathrm{rank}(G) \geq 3$ (see Section 9.4):

$$D(G) \leq d^2 \ln 2 + (1 - \ln 12)d - 1 + \ln 6. \tag{9.1}$$

Using some further analysis, we manage to improve this upper bound to the following when G is not a p-group and $\mathrm{rank}(G) \geq 3$ (see Section 9.4):

$$D(G) \leq (d-1)\left(1 + \ln\left(\frac{\left(\frac{d}{d-4}+1\right)^{d-4}}{6}\right)\right). \tag{9.2}$$

Remark 9.1.1. A tedious calculation shows that the bound in (9.2) is strictly less than the bound in (9.1) when $d \geq 5$ (if $d \leq 4$ then G is a p-group - see Lemma 9.3.3).

Under a specific condition on the representation of G as a direct sum of cyclic groups of prime power, we find that we can improve the bound even further to the following when G is not a p-group and $\mathrm{rank}(G) \geq 3$ (see Section 9.5):

$$D(G) \leq \begin{cases} 6d\ln 2 + 6 - 30\ln 2 & \text{if } \exp(G) \leq 6, \\ d^2\ln 2 + (1+\ln(5/3584))d - 1 + \ln(1792/5) & \text{otherwise.} \end{cases} \tag{9.3}$$

Remark 9.1.2. It is simple to check that the bounds in (9.3) are an improvement on the bound in (9.2) when $d \leq 11$. Notice that the second bound in (9.3) is strictly greater than the first bound when $d \geq 12$. For this reason, it is enough to show that the second bound in (9.3) is strictly less than the bound in (9.2) in order to show the improvement when $d \geq 12$. Another tedious calculation shows the latter.

We conjecture that the above bounds on $D(G)$ can be improved further. The following conjecture (deduced by combining Proposition 6.2.2 in [14] and a conjecture of Władysław Narkiewicz and Jan Śliwa stated in the last paragraph of [21]) is presented in [13].

Conjecture 9.1.3 ([13])**.** *For all finite abelian groups G, we have*

$$D(G) \leq d + r,$$

where $d := d^*(G)$ and $r := \mathrm{rank}(G)$.

Recall from Remark 1.1.14 that $d \geq \mathrm{rank}(G)$. Hence, if Conjecture 9.1.3 holds, then $D(G) \leq 2d$. We conjecture the following:

Conjecture 9.1.4. *For all finite abelian groups G, we have*

$$D(G) \leq 2d,$$

where $d := d^*(G)$.

Roughly speaking, the strategy we employ to find upper bounds on $D(G)$ in terms of d is to bound $D(G)$ from above by an increasing function of $|G|$ and then bound $|G|$ from above by a function of d. We shall find upper bounds on $|G|$ in terms of d in sections 9.2 and 9.3.

9.2 An elementary upper bound on $|G|$

There do not exist upper bounds on $|G|$ in terms of d in the literature, to the author's knowledge. In this section, we find prove the following elementary upper bound on $|G|$ in terms of d.

Theorem 9.2.1. *For all finite abelian groups G, we have*

$$|G| \leq 2^d,$$

where $d := d^*(G)$.

Before we proceed to proving Theorem 9.2.1, let us present a simple corollary and its proof.

Corollary 9.2.2. *For all finite abelian groups G, we have*

$$D(G) \leq 2^d,$$

where $d := d^*(G)$.

Proof. Combining Lemma 1.1.9 and Theorem 9.2.1, we find that

$$D(G) \leq |G| \leq 2^d. \qquad \square$$

We obtain Theorem 9.2.1 as a corollary of a stronger result for which require the following definitions.

Definition 9.2.3. Given $d \in \mathbb{N}$, we define

- $\Omega_d := \{G \mid d^*(G) = d\}$,
- $m_d := \max_{G \in \Omega_d} |G|$,
- $\mathcal{M}_d := \{G \in \Omega_d \mid |G| = m_d\}$.

Given $d \in \mathbb{N}$, the integer m_d represents the maximal order of a group H with $d^*(H) = d$. We are interested in finding m_d for an arbitrary d. More strongly, we are interested in finding the set of all groups H with $d^*(H) = d$ which achieve this maximal order, namely the set $\mathcal{M}_d$. We determine $\mathcal{M}_d$ in the following theorem from which we can easily deduce the proof of Theorem 9.2.1.

Theorem 9.2.4. *Given $d \in \mathbb{N}$, we have $\mathcal{M}_d = \{\mathbb{Z}_2^d\}$ (up to isomorphism).*

For the proof of Theorem 9.2.4 we need the following two results.

Theorem 9.2.5 (AM-GM inequality, see page 151 in [17])**.** *Let $x_1, \ldots, x_n$ be n arbitrary positive real numbers. Then*

$$(x_1 + \cdots + x_n)/n \geq (x_1 \cdots x_n)^{1/n}$$

with equality if and only if $x_1 = \cdots = x_n$.

Lemma 9.2.6. *Given* $d \in \mathbb{N}$, *the function*

$$\begin{aligned} f_d : \mathbb{N} &\longrightarrow \mathbb{R} \\ x &\mapsto ((d+x)/x)^x \end{aligned}$$

is strictly increasing.

Proof. It is sufficient to show $f_d(x) < f_d(x+1)$ for all $x \in \mathbb{N}$. Let $x \in \mathbb{N}$ and note that

$$\begin{aligned} & f_d(x) < f_d(x+1) \\ \iff & ((d+x)/x)^x < ((d+x+1)/(x+1))^{x+1} \\ \iff & ((d+x)/x)^x)^{1/(x+1)} < (d+x+1)/(x+1) = (1+x(d+x)/x)/(x+1). \end{aligned}$$

We can deduce the last strict inequality by applying Theorem 9.2.5 to the set of $x+1$ positive real numbers consisting of 1 and x copies of $(d+x)/x$ and noting that $d \neq 0$ implies $1 \neq (d+x)/x$. □

Proof of Theorem 9.2.4. Let $G \in \mathcal{M}_d$. Note that $\mathbb{Z}_2^d \in \Omega_d$ hence

$$2^d = |\mathbb{Z}_2^d| \leq m_d = |G|. \tag{9.4}$$

Let $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ for some $1 < n_1 \mid \cdots \mid n_r$. Then, using Theorem 9.2.5 and noting that $r \leq d$ (Remark 1.1.14), we find that

$$|G| = n_1 \cdots n_r \leq ((n_1 + \cdots + n_r)/r)^r = ((d+r)/r)^r = f_d(r) \leq f_d(d) = 2^d, \tag{9.5}$$

where f_d is the increasing function defined in Lemma 9.2.6. Hence, combining the inequalities (9.4) and (9.5) we find that

$$f_d(r) = f_d(d).$$

Therefore, since f_d is strictly increasing we have $r = d$, and so by Remark 1.1.14 we have $G \cong \mathbb{Z}_2^d$. This proves the theorem. $\square$

Proof of Theorem 9.2.1. We have $G \in \Omega_d$ and therefore

$$|G| \leq m_d = |\mathbb{Z}_2^d| = 2^d. \qquad \square$$

9.3 An improved upper bound on $|G|$

In this section we prove the following improved upper bound on $|G|$ when G is not a p-group.

Theorem 9.3.1. *For all finite abelian groups G which are not p-groups, we have*

$$|G| \leq (d/(d-4)+1)^{d-4},$$

where $d := d^*(G)$.

Remark 9.3.2. To see why Theorem 9.3.1 is an improvement on Theorem 9.2.1 when G is not a p-group, observe that for all $d \in \mathbb{N}$ we have

$$(d/(d-4)+1)^{d-4} = f_d(d-4) < f_d(d) = 2^d,$$

where f_d is as defined in Lemma 9.2.6.

We shall deduce Theorem 9.3.1 from the following lemma.

Lemma 9.3.3. *For all finite abelian groups G which are not p-groups, we have*

$$d \geq r+4,$$

where $d := d^*(G)$ and $r := \mathrm{rank}(G)$. In particular, $d \geq 5$.

Proof. Let $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ for some $1 < n_1 \mid \cdots \mid n_r$. Suppose $d \leq r+3$. Then $2r+3 \geq n_1 + \cdots + n_r$. If $n_i \geq 6$ for some i then we have that

$n_1 + \cdots + n_r \geq 2(r-1) + 6 = 2r + 4$. Hence $n_i \leq 5$ for all i. If $n_i = 3$ for some i then as $n_j \mid n_{j+1}$ for all j, we must have $n_i = 3$ for all i, and hence G is a 3-group. Similarly, if $n_i = 5$ for some i then G is a 5-group. If $n_i \in \{2, 4\}$ for some i then G is a 2-group. So we obtain a contradiction to the assumption that G is not a p-group in all cases which means that $d \geq r + 4$. □

Proof of Theorem 9.3.1. Define $r := \mathrm{rank}(G)$. By Lemma 9.3.3, we have $r \leq d - 4$. Therefore, as in the inequality (9.5), we obtain

$$|G| \leq f_d(r) \leq f_d(d-4) = (d/(d-4)+1)^{d-4},$$

where f_d is the increasing function defined in Lemma 9.2.6. □

9.4 A polynomial upper bound on $D(G)$

In this section we prove the following two theorems, the first of which utilises the general elementary upper bound $|G| \leq 2^d$, and the other which utilises the more refined upper bound $|G| \leq (d/(d-4)+1)^{d-4}$ when G is not a p-group.

Theorem 9.4.1. *For all finite abelian groups G with $\mathrm{rank}(G) \geq 3$ which are not p-groups, we have*

$$D(G) \leq d^2 \ln 2 + (1 - \ln 12)d - 1 + \ln 6,$$

where $d := d^*(G)$.

Theorem 9.4.2. *For all finite abelian groups G with $\mathrm{rank}(G) \geq 3$ which are not p-groups, we have*

$$D(G) \leq (d-1)\left(1 + \ln\left(\frac{\left(\frac{d}{d-4}+1\right)^{d-4}}{6}\right)\right),$$

where $d := d^*(G)$.

Remark 9.4.3. It is obvious that the polynomial bound in Theorem 9.4.1 is better than the two exponential bounds $D(G) \leq 2^d$ and $D(G) \leq |G| \leq (d/(d-4)+1)^{d-4}$.

In Section 9.2, we used the following strategy to obtain an upper bound on $D(G)$ in terms of d: bound $D(G)$ from above by the trivial upper bound $|G|$, and then bound $|G|$ from above by a function of d. This strategy leads to the question of whether there exist upper bounds on $D(G)$ in terms of $|G|$ that would lead to an improved upper bound on $D(G)$ in terms of d. The answer to this question is yes. In order to establish the improved upper bounds in Theorem 9.4.1 and Theorem 9.4.2, we bound $D(G)$ from above by the following upper bound on $D(G)$ in terms of $|G|$ proved by Boas and Kruyswijk.

Theorem 9.4.4 (Theorem 7.1 in [5])**.** *For all finite abelian groups G, we have*

$$D(G) \leq n\left(1+\ln\frac{|G|}{n}\right),$$

where $n := \exp(G)$.

Proof of Theorem 9.4.1. Define $n := \exp(G)$. Note that, since G is not a p-group, we have $n \geq 6$. Hence, the upper bound $|G| \leq 2^d$ from Theorem 9.2.1 implies that $\frac{|G|}{n} \leq \frac{2^d}{6}$. This implies that

$$n\left(1+\ln\frac{|G|}{n}\right) \leq n\left(1+\ln\frac{2^d}{6}\right).$$

Now note that, since $\operatorname{rank}(G) \geq 3$, we have $n \leq d-1$. Indeed, if $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ for some $1 < n_1 \mid \cdots \mid n_r$, then $n = d - n_1 - \cdots - n_{r-1} + r$. Now, $n_1 + \cdots + n_{r-1} \geq 2(r-1)$, hence $n \leq d-r+2$. Therefore, since $\operatorname{rank}(G) \geq 3$, we deduce that $n \leq d-1$. Consequently, using Theorem 9.4.4, we obtain

$$D(G) \leq (d-1)\left(1 + \ln \frac{2^d}{6}\right) = d^2 \ln 2 + (1 - \ln 12)d - 1 + \ln 6.$$

□

Proof of Theorem 9.4.2. Define $n := \exp(G)$. Similar to the proof of Theorem 9.4.1, noting that $6 \leq n \leq d-1$ and using the bound in Theorem 9.4.4 with the upper bound $|G| \leq (d/(d-4)+1)^{d-4}$ from Theorem 9.3.1, we find that

$$D(G) \leq n\left(1 + \ln \frac{|G|}{n}\right) \leq (d-1)\left(1 + \ln\left(\frac{(d/(d-4)+1)^{d-4}}{6}\right)\right).$$

□

9.5 Special polynomial upper bounds on $D(G)$

The main theorem of this section requires the following preliminaries.

Theorem 9.5.1 (Theorem 3 on page 48 in [26]). *For any non-trivial finite abelian group G, there exist prime numbers $p_1, \ldots, p_t$ (not necessarily distinct) and $l_1, \ldots, l_t \in \mathbb{N}$ such that*

$$G \cong \mathbb{Z}_{p_1^{l_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{l_t}}.$$

This representation of G is unique (up to re-ordering of the summands).

Definition 9.5.2. For $G \cong \mathbb{Z}_{p_1^{l_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{l_t}}$ with $p_1, \ldots, p_t$ primes, define

- $\mathrm{Div}_G := \{p \mid p_i = p_j = p \text{ for some distinct } i, j \in \{1, \ldots, t\}\}$;
- $p^G_{\max} := \max \mathrm{Div}_G$ (given that $\mathrm{Div}_G \neq \emptyset$).

Remark 9.5.3. If $G \cong \mathbb{Z}_{p_1^{l_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{l_t}}$ with $p_1, \ldots, p_t$ distinct primes then G is cyclic. Hence, for non-cyclic G we always have $\mathrm{Div}_G \neq \emptyset$.

The main theorem of this section is the following:

Theorem 9.5.4. *For all finite abelian groups G with* $\mathrm{rank}(G) \geq 3$ *and* $p^G_{\max} = 2$ *which are not p-groups, we have*

$$D(G) \leq \begin{cases} 6d\ln 2 + 6 - 30\ln 2 & \text{if } \exp(G) \leq 6, \\ d^2 \ln 2 + (1 + \ln(5/3584))d - 1 + \ln(1792/5) & \text{otherwise}, \end{cases}$$

where $d := d^*(G)$.

We obtain Theorem 9.5.4 by improving on the upper bound on $|G|$ given in Theorem 9.3.1 in the case when $p^G_{\max} = 2$, as follows.

Theorem 9.5.5. *For all finite abelian groups G which are not p-groups and for which $p^G_{\max} = 2$, we have*

$$|G| \leq \begin{cases} 6 \cdot 2^{d-5} & \text{if } \exp(G) \leq 6, \\ \frac{5}{8} \cdot 2^{d-5} & \text{otherwise}, \end{cases}$$

where $d := d^*(G)$.

Proof. Let $p_1, \dots, p_t$ be primes such that

$$G \cong \mathbb{Z}_{p_1^{l_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{l_t}} .$$

Since $p^G_{\max} = 2$ and G is not a p-group, there exists r with $t > r \geq 2$ such that without loss of generality we may assume the following: $p_i = 2$ for all $i \in [1, r]$; $p_i \geq 3$ for all $i \in [r+1, t]$; and p_i are pairwise distinct for all $i \in [r+1, t]$. Hence we can write

$$G \cong \mathbb{Z}_{2^{l_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{l_{r-1}}} \oplus \mathbb{Z}_{2^{l_r}h},$$

where $h := p_{r+1}^{l+1} \cdots p_t^{l_t} \geq 3$ is odd and $1 \leq l_1 \leq \cdots \leq l_r$. Further, we can define $n_r := 2^{l_r}h$ and write

$$G \cong \mathbb{Z}_2^{\beta_1} \oplus \mathbb{Z}_{2^2}^{\beta_2} \oplus \cdots \oplus \mathbb{Z}_{2^{l_r}}^{\beta_{l_r}} \oplus \mathbb{Z}_{n_r}$$

for some $\beta_i \geq 0$ such that $\sum_i \beta_i = r - 1$. Hence,

$$d = n_r - 1 + \sum_i \beta_i(2^i - 1) = n_r - 1 + \sum_i \beta_i(2^i - 2) + r - 1.$$

Moreover,

$$|G| = 2^{\sum_i i\beta_i} n_r = 2^{r-1+\sum_i \beta_i(i-1)} n_r.$$

Furthermore,

$$|G| = 2^{d-n_r+1-\sum_i \beta_i(2^i-2)+\sum_i \beta_i(i-1)} n_r = 2^{d-n_r+1-\sum_i \beta_i(2^i-1-i)} n_r.$$

Therefore,

$$|G| \leq \frac{5}{8}.2^{d-5}$$
$$\iff n_r \leq \frac{5}{8}.2^{n_r-6+\sum_i \beta_i(2^i-1-i)}.$$

We claim that if $n_r > 6$ then $n_r \leq \frac{5}{8}.2^{n_r-6}$. This claim can be easily proved by induction on n_r whilst noting that if $n_r = 2^{l_r}.h > 6$ then in fact $n_r \geq 10$ (since $2 \nmid 7$, since 8 does not contain an odd divisor strictly greater than 2, and since $2 \nmid 9$). Using this claim and the fact that $2^i \geq i + 1$ for all $i \in \mathbb{N}$, we deduce that

$$\frac{5}{8}.2^{n_r-6+\sum_i \beta_i(2^i-1-i)} \geq \frac{5}{8}.2^{n_r-6} \geq n_r$$

when $n_r > 6$. This proves the second bound in the theorem. Similarly,

$$|G| \leq 6.2^{d-5}$$
$$\iff n_r \leq 6.2^{n_r-6+\sum_i \beta_i(2^i-1-i)}$$

which clearly holds if $n_r = 6$. Noting that $n_r = 2^{l_r}h \geq 6$, the first bound in the theorem is also proved. $\square$

Remark 9.5.6. It is simple to see that, in the case when G is not a p-group and $p^G_{\max} = 2$, the upper bounds on $|G|$ in Theorem 9.5.5 are an improvement on the upper bound on $|G|$ presented in Theorem 9.3.1. Indeed, for all integers $d \geq 5$ we have,

$$\begin{aligned} & 6.2^{d-5} \leq (d/(d-4)+1)^{d-4} \\ \iff & 3.2^{d-4} \leq 2^{d-4}((d-2)/(d-4))^{d-4} \\ \iff & 3^{1/(d-4)} \leq (d-2)/(d-4), \end{aligned}$$

where the last inequality can be deduced by applying Theorem 9.2.5 to the multiset of $d-4$ positive real numbers consisting of 3 and $d-5$ copies of 1.

Proof of Theorem 9.5.4. Define $n := \exp(G)$. Suppose $n \leq 6$. Then in fact $n = 6$. Hence, using the bound in Theorem 9.4.4 with the upper bound in Theorem 9.5.5, we find that

$$D(G) \leq n\left(1 + \ln\frac{|G|}{n}\right) \leq 6(1 + \ln 2^{d-5}) = 6d\ln 2 + 6 - 30\ln 2.$$

Suppose $n \geq 7$. Then the upper bound in Theorem 9.5.5 along with the property $n \leq d-1$, implies that

$$D(G) \leq n\left(1 + \ln\frac{|G|}{n}\right) \leq (d-1)\left(1 + \ln\frac{5 \cdot 2^{d-8}}{7}\right).$$

Simplifying the right hand side of the above inequality, we obtain the required bound. $\square$

Chapter 10

Open problems

In this chapter we present some open problems that seem to arise naturally from the content in the rest of the thesis. It is obvious that proving or disproving Conjecture 1.2.3, Conjecture 2.1.6, or Conjecture 9.1.3 would be a great milestone in the subject area. We list some other (possibly easier) open problems in the subject area that may give rise to ideas/techniques that could be useful in proving or finding counterexamples for these conjectures.

Given Theorem 6.1, the smallest abelian group of rank 3 for which the Davenport constant is unknown, now becomes $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{15}$.

Problem 10.1. Find the Davenport constant of the group $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{15}$.

Remark 10.2. A theoretical approach to Problem 10.1 leaves us with a number of open subcases. We have tried to deal with these subcases using a computer assisted approach similar to the one used for the proof of Theorem 6.1, however we find that our algorithms are not terminating in a feasible amount of time.

Problem 10.3. Given an arbitrary positive integer d, find the Davenport constant of the group $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5d}$.

Problem 10.4. Given an arbitrary prime number p, find the Davenport constant of the group $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{2p}$.

Problem 10.5. Given an arbitrary finite abelian group G, find an upper bound on $D(G)$ which is a linear polynomial in $d^*(G)$.

Problem 10.6. Prove or disprove Conjecture 9.1.4.

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